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ON SOME COEFFICIENT INEQUALITY IN THE SUBCLASS OF CLOSE-TO-CONVEX FUNCTIONS

Summary

A coefficient inequality related to the Fekete-Szegö-Goluzin problem in some subclass of close-to-convex functions is shown.

Keywords and phrases: Coefficient inequality, close-to-convex functions, Fekete-Szegö-Goluzin problem.

1. Introduction

To find for each $\lambda \in [0,1]$ the maximum value of the coefficient functional

$$\Phi_\lambda(f) := \left|a_3 - \lambda a_2^2\right|$$

over the class $\mathcal{S}$ of univalent functions f in the unit disk $\mathbf{D} := \{z \in \mathbf{C} : |z| < 1\}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbf{D}, \tag{1.1}$$

is a well known problem having the source in the paper [5] by Fekete and Szegö. They considered the case $\lambda := (k-1)/(2k)$, $k = 2, 3, \ldots$, however the case $\lambda \in (0,1)$ was first discussed and solved by Goluzin [6]. Particularly, recall that

$$\max_{f \in \mathcal{S}} \Phi_\lambda(f) = \begin{cases} 1 + 2\exp\left(-2\lambda/(1-\lambda)\right), & \lambda \in [0,1), \\ 1, & \lambda := 1. \end{cases}$$

The problem to find $\max_{f\in\mathcal{F}}\Phi_\lambda(f)$ over compact subclasses $\mathcal{F}$ of the class $\mathcal{A}$ of all analytic functions f in $\mathbf{D}$ of the form (1.1), as well as for λ being an arbitrary real or complex number, was studied by many authors (see e.g., [8], [12], [9], [10], [15], [32], [28], [17], [13], [11], [16], [2]).

Let $\mathcal{S}^*$ denote the class of functions $f\in\mathcal{A}$ such that

$$\operatorname{Re}\frac{zf'(z)}{f(z)}>0,\quad z\in\mathbf{D},$$

called *starlike*, and let $\mathcal{S}^c$ denote the class of functions $f\in\mathcal{A}$ such that

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>0,\quad z\in\mathbf{D},$$

called *convex*. Clearly, $\mathcal{S}^c\subsetneq\mathcal{S}^*$.

Given $\delta\in(-\pi/2,\pi/2)$ and $g\in\mathcal{S}^*$, let $\mathcal{C}_\delta(g)$ denote the class of functions $f\in\mathcal{A}$ such that

$$\operatorname{Re}\left\{\mathrm{e}^{\mathrm{i}\delta}\frac{zf'(z)}{g(z)}\right\}>0,\quad z\in\mathbf{D},$$

called *close-to-convex with argument δ with respect to g*. For $g\in\mathcal{S}^*$ let

$$\mathcal{C}(g):=\bigcup_{\delta\in(-\pi/2,\pi/2)}\mathcal{C}_\delta(g)$$

be the class of functions called *close-to-convex with respect to g*. For $\delta\in(-\pi/2,\pi/2)$ let

$$\mathcal{C}_\delta:=\bigcup_{g\in\mathcal{S}^*}\mathcal{C}_\delta(g),\quad \mathcal{C}_\delta^c:=\bigcup_{h\in\mathcal{S}^c}\mathcal{C}_\delta(h).$$

Let

$$\mathcal{C}:=\bigcup_{\delta\in(-\pi/2,\pi/2)}\bigcup_{g\in\mathcal{S}^*}\mathcal{C}_\delta(g)$$

denote the class of *close-to-convex* functions (see [27, pp. 184-185], [14]), and let

$$\mathcal{C}^c:=\bigcup_{\delta\in(-\pi/2,\pi/2)}\bigcup_{h\in\mathcal{S}^c}\mathcal{C}_\delta(h).$$

In [15] Keogh and Merkes proved that

$$\max_{f\in\mathcal{C}_0}\Phi_\lambda(f)=\begin{cases}|3-4\lambda|, & \lambda\in\mathbf{R}\setminus(1/3,1),\\ 1/3+4/(9\lambda), & \lambda\in[1/3,2/3],\\ 1, & \lambda\in[2/3,1].\end{cases}$$

For $\lambda\in[0,1]$ Koepf [17] extended the above result for the whole class $\mathcal{C}$ showing that

$$\max_{f\in\mathcal{C}}\Phi_\lambda(f)=\max_{f\in\mathcal{C}_0}\Phi_\lambda(f).$$

In fact, the last result holds for all real λ.

For the class $\mathcal{C}_0^c$ the Fekete-Szegö problem was considered by Abdel-Gawad and Thomas [1]. For the whole class $\mathcal{C}^c$ the computing was done by Srivastava, Mishra and Das [31]. Their results together with some remark of [22] can be written as follow:

$$\max_{f\in\mathcal{C}^c}\Phi_\lambda(f)=\max_{f\in\mathcal{C}_0^c}\Phi_\lambda(f)=\begin{cases}5/3-9\lambda/4, & \lambda\in[0,2/9],\\ 2/3+1/(9\lambda), & \lambda\in[2/9,2/3],\end{cases}$$

and

$$\max_{f\in\mathcal{C}_0^c}\Phi_\lambda(f)\le\max_{f\in\mathcal{C}^c}\Phi_\lambda(f)\le\frac{5}{6},\quad \lambda\in(2/3,1].$$

Given $\alpha\in[0,1]$ let, for $z\in\mathbf{D}$,

$$g_\alpha(z):=\frac{z}{(1-\alpha z)^2},\quad h_\alpha(z):=\frac{z}{1-\alpha z}.$$

Clearly, $g_\alpha\in\mathcal{S}^*$ and $h_\alpha\in\mathcal{S}^c$ for $\alpha\in[0,1]$. The corresponding classes $\mathcal{C}(g_\alpha)$ and $\mathcal{C}(h_\alpha)$ are defined, respectively, as: for $\delta\in(-\pi/2,\pi/2)$,

$$\operatorname{Re}\left\{e^{i\delta}(1-\alpha z)^2f'(z)\right\}>0,\quad z\in\mathbf{D}, \tag{1.2}$$

and

$$\operatorname{Re}\left\{e^{i\delta}(1-\alpha z)f'(z)\right\}>0,\quad z\in\mathbf{D}. \tag{1.3}$$

For the class $\mathcal{C}(g_\alpha)$ it was shown in [18] that

$$\max_{f\in\mathcal{C}(g_\alpha)}\Phi_\lambda(f) \tag{1.4}$$

$$\le\begin{cases}\left|\frac{2}{3}+\frac{4}{3}\alpha+\alpha^2-(1+\alpha)^2\lambda\right|, & \lambda\in\mathbf{R}\setminus(\tau_1(\alpha),\tau_2(\alpha)),\\ \alpha^2\left(\frac{(2-3\lambda)^2}{3(2-|2-3\lambda|)}+|1-\lambda|\right)+\frac{2}{3}, & \lambda\in[\tau_1(\alpha),\tau_2(\alpha)],\end{cases}$$

where

$$\tau_1(\alpha):=\frac{2\alpha}{3(1+\alpha)},\quad \tau_2(\alpha):=\frac{2(2+\alpha)}{3(1+\alpha)}. \tag{1.5}$$

The sharpness holds for each $\alpha\in(0,1]$ and each $\lambda\in\mathbf{R}\setminus(2/3,\tau_2(\alpha))$ as well as for $\alpha:=0$ and each $\lambda\in\mathbf{R}$.

As it is known, the Koebe function $k:=g_1$ ($\alpha:=1$) is extremal for various computational problems in the class $\mathcal{S}^*$ of starlike functions. Moreover the class $\mathcal{C}_0(k)$ of functions called *convex in the positive direction of the real axis* plays an important role as the subclass of functions convex in one direction defined by Robertson [30] and it was intensively recently studied (see e.g., [3], [24], [4]). For the class $\mathcal{C}(k)$ the Fekete-Szegö problem was separately considered in [19] where it was shown that

$$\max_{f\in\mathcal{C}(k)\cup\{k_0\}}\Phi_\lambda(f)=\max_{f\in\mathcal{C}}\Phi_\lambda(f),\quad \lambda\in\mathbf{R},$$

where

$$k_0(z):=\frac{z}{1-z^2},\quad z\in\mathbf{D},$$

is the odd close-to-convex function.

For the class $\mathcal{C}(h_\alpha)$ it was shown in [21] that

$$\tag{1.6} \max_{f\in\mathcal{C}(h_\alpha)} \Phi_\lambda(f) \le \begin{cases} \alpha^2\left|\frac{1}{3}-\frac{\lambda}{4}\right| + (1+\alpha)\left|\frac{2}{3}-\lambda\right|, & \lambda\in\mathbf{R}\setminus[\tau_1'(\alpha),\tau_2'(\alpha)],\\ \alpha^2\left(\frac{(2-3\lambda)^2}{12(2-|2-3\lambda|)} + \left|\frac{1}{3}-\frac{\lambda}{4}\right|\right) + \frac{2}{3}, & \lambda\in[\tau_1'(\alpha),\tau_2'(\alpha)],\end{cases}$$

where

$$\tau_1'(\alpha) := \frac{2\alpha}{3(2+\alpha)}, \quad \tau_2'(\alpha) := \frac{2(4+\alpha)}{3(2+\alpha)}.$$

The sharpness holds for each $\alpha\in(0,1]$ and each $\lambda\in\mathbf{R}\setminus(2/3,4/3)$, as well as for $\alpha:=0$ and each $\lambda\in\mathbf{R}$.

As it is known, the function $h := h_1$ ($\alpha := 1$) is extremal for computational problems in the class $\mathcal{S}^c$ of convex functions. For the first time the inequality (1.3) with $\alpha = 1$, treated as the univalence criterium, was distinguished explicitly in [27, p. 185]. For the class $\mathcal{C}(h)$ the Fekete-Szegö problem was separately considered in [20] where it was shown (1.6) for $\alpha = 1$; particularly, it was proved that for $\lambda\in[0,2/3]$,

$$\max_{f\in\mathcal{C}(h)} \Phi_\lambda(f) = \max_{f\in\mathcal{C}_0^c} \Phi_\lambda(f) = \max_{f\in\mathcal{C}^c} \Phi_\lambda(f).$$

For $\alpha := 0$ the conditions (1.2) and (1.3) reduce to

$$\tag{1.7} \operatorname{Re}\left\{e^{i\delta} f'(z)\right\} > 0, \quad z\in\mathbf{D}.$$

Functions f having such a property are called *of bounded turning with argument* δ and form the class denoted usually as $\mathcal{P}'(\delta)$, and further the class $\mathcal{P}' := \mathcal{C}(g_0)$ of functions called *of bounded turning* (cf. [7, Vol. I, p. 101]). On the other hand, the condition (1.7) is known as a famous criterium of univalence due to Noshiro [26] and Warschawski [33]. By setting $\alpha := 0$ into (1.4) or (1.6) we get the following result published, among other results, in [13, Theorem 2.3]: for $\lambda\in[0,4/3]$,

$$\max_{f\in\mathcal{P}'} \Phi_\lambda(f) = \frac{2}{3}.$$

In this paper we unify results recalled above for the classes $\mathcal{C}(g_\alpha)$ and $\mathcal{C}(h_\alpha)$ with $\alpha\in[0,1]$. Given $\alpha,\beta\in[0,1]$, let

$$g_{\alpha,\beta}(z) := \frac{z}{(1-\alpha z)(1-\beta z)}, \quad z\in\mathbf{D}.$$

Thus the class $\mathcal{C}(g_{\alpha,\beta})$ is defined as

$$\operatorname{Re}\left\{e^{i\delta}(1-\alpha z)(1-\beta z) f'(z)\right\} > 0, \quad z\in\mathbf{D}.$$

Clearly, $\mathcal{C}(g_{\alpha,\alpha}) = \mathcal{C}(g_\alpha)$ and $\mathcal{C}(g_{\alpha,0}) = \mathcal{C}(g_{0,\alpha}) = \mathcal{C}(h_\alpha)$ for $\alpha\in[0,1]$. The class $\mathcal{C}(g_{\alpha,\beta})$ appeared in [23] and [25] as a generalization of convexity in one direction (see [30]). In the main result we show the upper bound for the Fekete-Szegö functional for the class $\mathcal{C}(g_{\alpha,\beta})$ generalizing (1.4) and (1.6).

2. Main result

Let $\mathcal{P}$ denote the class of analytic functions in $\mathbf{D}$ of the form

$$p(z) := 1 + \sum_{n=1}^{\infty} c_n z^n, \quad z \in \mathbf{D}, \tag{2.1}$$

having a positive real part in $\mathbf{D}$. Let

$$L(z) := \frac{1+z}{1-z}, \quad z \in \mathbf{D}.$$

Lemma 2.1. ([25, pp. 41,46]) *If $p \in \mathcal{P}$ is of the form* (2.1)*, then*

$$|c_n| \le 2, \quad n \in \mathbf{N}, \tag{2.2}$$

and

$$\left|c_2 - c_1^2/2\right| \le 2 - |c_1|^2/2. \tag{2.3}$$

Both inequalities are sharp. The equality in (2.2) *holds for L and in* (2.3) *for every function*

$$p_{t,\theta}(z) := tL\left(\mathrm{e}^{\mathrm{i}\theta} z\right) + (1-t)L\left(\mathrm{e}^{2\mathrm{i}\theta} z^2\right) = 1 + 2t\mathrm{e}^{\mathrm{i}\theta} z + 2\mathrm{e}^{2\mathrm{i}\theta} z^2 + \ldots, \quad z \in \mathbf{D}, \tag{2.4}$$

where $t \in [0,1]$ and $\theta \in \mathbf{R}$.

The details of the proof of the main theorem are almost exactly the same as of Theorem 2.4 of [18]. Therefore here we present only a sketch of the proof. Similar method of proof with all details of computing appeared in [21]. The main theorem of the paper is

Theorem 2.2. *Let $\alpha, \beta \in [0,1]$. Then*

$$\max_{f \in \mathcal{C}(g_{\alpha,\beta})} \Phi_\lambda(f) \tag{2.5}$$

$$\le \begin{cases} \dfrac{1}{6}\left|\alpha^2 + \beta^2 + (\alpha+\beta)^2\left(1 - \dfrac{3}{2}\lambda\right)\right| \\ \quad + (1+\alpha+\beta)\left|\dfrac{2}{3} - \lambda\right|, & \lambda \in \mathbf{R} \setminus (\tau_1(\alpha,\beta), \tau_2(\alpha,\beta)) \\ \dfrac{1}{6}\left|\alpha^2 + \beta^2 + (\alpha+\beta)^2\left(1 - \dfrac{3}{2}\lambda\right)\right| \\ \quad + \dfrac{(\alpha+\beta)^2(2-3\lambda)^2}{12(2-|2-3\lambda|)} + \dfrac{2}{3}, & \lambda \in [\tau_1(\alpha,\beta), \tau_2(\alpha,\beta)], \end{cases}$$

where

$$\tau_1(\alpha,\beta) := \frac{2(\alpha+\beta)}{3(2+\alpha+\beta)}, \; \tau_2(\alpha,\beta) := \frac{2(4+\alpha+\beta)}{3(2+\alpha+\beta)}.$$

For each $\alpha, \beta \in [0,1]$, $(\alpha,\beta) \ne (0,0)$, and each $\lambda \in \mathbf{R} \setminus (2/3, \lambda(\alpha,\beta))$, where

$$\lambda(\alpha,\beta) := \frac{2}{3} + \frac{2}{3}\max\left\{\frac{\alpha^2+\beta^2}{(\alpha+\beta)^2}, \frac{2}{2+\alpha+\beta}\right\},$$

as well as for $\alpha = \beta := 0$ *and each* $\lambda \in \mathbf{R}$*, the inequality is sharp and the equality is attained by a function in* $\mathcal{C}_0(g_{\alpha,\beta})$*. In particular,*

(i) *when* $\alpha, \beta \in [0,1]$, $(\alpha,\beta) \neq (0,0)$, *then for each* $\lambda \in [\tau_1(\alpha,\beta), 2/3]$ *the second equality in* (2.5) *is attained by a function* $f_{\alpha,\beta,t_{\alpha,\beta,\lambda}}$ *such that*

$$f'_{\alpha,\beta,t_{\alpha,\beta,\lambda}}(z) = \frac{p_{t_{\alpha,\beta,\lambda},0}(z)}{(1-\alpha z)(1-\beta z)}, \quad z \in \mathbf{D}, \tag{2.6}$$

with $f_{\alpha,\beta,t_{\alpha,\beta,\lambda}}(0) := 0$ *and*

$$t_{\alpha,\beta,\lambda} := (\alpha+\beta)\,(1/(3\lambda) - 1/2)\,;$$

(ii) *when* $\alpha, \beta \in [0,1]$, $(\alpha,\beta) \neq (0,0)$, *then for each* $\lambda \in \mathbf{R} \setminus (\tau_1(\alpha,\beta), \lambda(\alpha,\beta))$ *the first equality in* (2.5) *is attained by the function* $f_{\alpha,\beta,1}$, *given by* (2.6) *with* $t_{\alpha,\beta,\lambda} := 1$, *i.e., when* $\alpha, \beta \in (0,1)$, $\alpha \neq \beta$, *by the function:*

$$f_{\alpha,\beta,1}(z) := \frac{1}{\alpha-\beta}\left(\frac{1+\alpha}{\alpha(1-\alpha)}\log(1-\alpha z) - \frac{1+\beta}{\beta(1-\beta)}\log(1-\beta z)\right) - \frac{2}{(1-\alpha)(1-\beta)}\log(1-z), \quad z \in \mathbf{D},\ \log 1 := 0; \tag{2.7}$$

when $\beta := 1$, $\alpha \in [0,1)$, *by the function*

$$f_{\alpha,1,1}(z) := \frac{1}{1-\alpha}\left(\frac{1+\alpha}{1-\alpha}\log\frac{1-z}{1-\alpha z} + \frac{2z}{1-z}\right), \quad z \in \mathbf{D},\ \log 1 := 0; \tag{2.8}$$

when $\alpha := 1$, $\beta \in [0,1)$, *by the function* $f_{1,\beta,1} := f_{\beta,1,1}$; *when* $\beta := 0$ *and* $\alpha \in (0,1)$ *by the function*

$$f_{\alpha,0,1}(z) := \frac{1}{1-\alpha}\left(\frac{1+\alpha}{\alpha}\log(1-\alpha z) - 2\log(1-z)\right), \quad z \in \mathbf{D},\ \log 1 := 0; \tag{2.9}$$

when $\alpha := 0$ *and* $\beta \in (0,1)$ *by the function* $f_{0,\beta,1} := f_{\beta,0,1}$; *when* $\beta := \alpha \in (0,1)$ *by the function*

$$f_{\alpha,\alpha,1}(z) := \frac{1}{(1-\alpha)^2}\log\frac{1-\alpha z}{1-z} - \frac{1+\alpha}{1-\alpha}\cdot\frac{z}{1-\alpha z}, \quad z \in \mathbf{D},\ \log 1 := 0; \tag{2.10}$$

when $\beta = \alpha := 1$ *by the Koebe function* $f_{1,1,1} := k$.

(iii) *when* $\beta = \alpha := 0$, *then for each* $\lambda \in [0, 4/3]$ *the second equality in* (2.5) *is attained by the function*

$$f_{0,0,0}(z) := -z + \log\frac{1+z}{1-z}, \quad z \in \mathbf{D},\ \log 1 := 0; \tag{2.11}$$

for each $\lambda \in \mathbf{R} \setminus (0, 4/3)$ *the first equality in* (2.5) *is attained by the function*

$$f_{0,0,1}(z) := -z - 2\log(1-z), \quad z \in \mathbf{D},\ \log 1 := 0. \tag{2.12}$$

Proof. Fix $\alpha, \beta \in [0,1]$. Observe that $f \in \mathcal{C}(g_{\alpha,\beta})$ if and only if for $z \in \mathbf{D}$,

$$zf'(z) = \mathrm{e}^{-\mathrm{i}\delta} g_{\alpha,\beta}(z)\,(p(z)\cos\delta + \mathrm{i}\sin\delta) \tag{2.13}$$

for $\delta \in (-\pi/2, \pi/2)$ and $p \in \mathcal{P}$. For $z \in \mathbf{D}$ we have

$$g_{\alpha,\beta}(z) = z + (\alpha+\beta)z^2 + (\alpha^2+\alpha\beta+\beta^2)z^3 + \dots$$

Setting the above series with the series (1.1) and (2.1) into (2.13) by comparing coefficients we get

$$\begin{aligned} a_2 &= \frac{1}{2}\left(c_1 \mathrm{e}^{-\mathrm{i}\delta}\cos\delta + \alpha + \beta\right), \\ a_3 &= \frac{1}{3}\left(c_2 \mathrm{e}^{-\mathrm{i}\delta}\cos\delta + (\alpha+\beta)c_1\mathrm{e}^{-\mathrm{i}\delta}\cos\delta + \alpha^2+\alpha\beta+\beta^2\right). \end{aligned} \tag{2.14}$$

Let $\lambda \in \mathbf{R}$. By (2.14) and (2.3) we have

$$\begin{aligned} \Phi_\lambda(f) = &\left|\frac{1}{3}(\alpha^2+\alpha\beta+\beta^2) - \frac{1}{4}(\alpha+\beta)^2\lambda + \frac{1}{3}\left(c_2 - \frac{c_1^2}{2}\right)\mathrm{e}^{-\mathrm{i}\delta}\cos\delta\right. \\ &\left. + \frac{c_1^2}{6}\left(1 - \frac{3}{2}\lambda\mathrm{e}^{-\mathrm{i}\delta}\cos\delta\right)\mathrm{e}^{-\mathrm{i}\delta}\cos\delta + \frac{1}{2}(\alpha+\beta)\left(\frac{2}{3}-\lambda\right)c_1\mathrm{e}^{-\mathrm{i}\delta}\cos\delta\right| \end{aligned}$$

$$\begin{aligned} &\le \left|\frac{1}{3}(\alpha^2+\alpha\beta+\beta^2) - \frac{1}{4}(\alpha+\beta)^2\lambda\right| + \frac{1}{3}\left(2 - \frac{|c_1|^2}{2}\right)\cos\delta \\ &\quad + \frac{|c_1|^2}{6}\left|1 - \frac{3}{2}\lambda\mathrm{e}^{-\mathrm{i}\delta}\cos\delta\right|\cos\delta + \frac{1}{2}(\alpha+\beta)\left|\frac{2}{3}-\lambda\right||c_1|\cos\delta \\ &= \frac{1}{6}\left|\alpha^2+\beta^2+\frac{1}{2}(\alpha+\beta)^2\gamma\right| + \frac{1}{6}\left(4 + x^2(s_\gamma(y)-1) + (\alpha+\beta)|\gamma|x\right)y, \end{aligned} \tag{2.15}$$

where $x := |c_1|$, $y := \cos\delta$, $\gamma := 2 - 3\lambda$ and

$$s_\gamma(y) := \sqrt{1 - (1-\gamma^2/4)\,y^2}.$$

In view of (2.2), $x \in [0,2]$ and clearly, $y \in (0,1]$. Set $\mu := (\alpha+\beta)/2$ and $R := [0,2]\times[0,1]$. Thus $\mu \in [0,1]$ and with $\gamma \in \mathbf{R}$ define

$$F_{\mu,\gamma}(x,y) := \left(4 + x^2(s_\gamma(y)-1) + 2\mu|\gamma|x\right)y, \quad (x,y) \in R.$$

Hence and by (2.15) we have

$$\max_{f\in\mathcal{C}(g_{\alpha,\beta})} \Phi_\lambda(f) \le \frac{1}{6}\left|\alpha^2+\beta^2+\frac{1}{2}(\alpha+\beta)^2\gamma\right| + \frac{1}{6}\max_{(x,y)\in R} F_{\mu,\gamma}(x,y). \tag{2.16}$$

Now for each $\mu \in [0,1]$ and $\gamma \in \mathbf{R}$ we find the maximum value of $F_{\mu,\gamma}$ on the rectangle R. Since from now the computing is exactly identical as in [21] we demonstrate the short sketch of the proof only.

In the corners of R we have

$$\begin{aligned} &F_{\mu,\gamma}(0,0) = F_{\mu,\gamma}(2,0) = 0, \\ &F_{\mu,\gamma}(0,1) = 4,\ F_{\mu,\gamma}(2,1) = 2(1+2\mu)|\gamma|. \end{aligned} \tag{2.17}$$

For $x := 0$ and $y \in (0,1)$ we have a linear function and for $x \in (0,2)$ and $y := 0$ we have a constant function.

For $x \in (0,2)$ and $y := 1$ we have a function

$$G_{\mu,\gamma}(x) := F_{\mu,\gamma}(x,1) = \left(\frac{|\gamma|}{2} - 1\right) x^2 + 2\mu|\gamma|x + 4$$

which for $|\gamma| = 2$ reduces to the linear function and for $|\gamma| \neq 2$ has the unique critical point at

$$x = \frac{2\mu|\gamma}{2-|\gamma|} =: x_{\mu,\gamma} \in (0,2)$$

if and only if

$$\mu \neq 0 \wedge 0 < |\gamma| < \frac{2}{1+\mu}. \tag{2.18}$$

Moreover

$$F_{\mu,\gamma}(x_{\mu,\gamma},1) = G_{\mu,\gamma}(x_{\mu,\gamma}) = \frac{2\mu^2\gamma^2}{2-|\gamma|} + 4. \tag{2.19}$$

For $x := 2$ and $y \in (0,1)$ we have a function

$$H_{\mu,\gamma}(y) := F_{\mu,\gamma}(2,y) = 4ys_\gamma(y) + 4\mu|\gamma|y$$

which for $|\gamma| = 2$ reduces to the linear function and for $|\gamma| \neq 2$ has the unique critical point at

$$y = \sqrt{\frac{4-\mu^2\gamma^2+\mu|\gamma|\sqrt{\mu^2\gamma^2+8}}{2(4-\gamma^2)}} =: y_{\mu,\gamma} \in (0,1)$$

if and only if

$$|\gamma| < \sqrt{\frac{2}{1+\mu}}. \tag{2.20}$$

Moreover

$$\begin{aligned} F_{\mu,\gamma}(2,y_{\mu,\gamma}) &= H_{\mu,\gamma}(y_{\mu,\gamma}) \\ &= \sqrt{\frac{4-\mu^2\gamma^2+\mu|\gamma|\sqrt{\mu^2\gamma^2+8}}{2(4-\gamma^2)}}\left(\sqrt{\mu^2\gamma^2+8}+3\mu|\gamma|\right). \end{aligned} \tag{2.21}$$

Repeating exactly argumentation of [18, pp. 8-10] we show that for each $\mu \in [0,1]$ and each $\gamma \in \mathbf{R}$ the function $F_{\mu,\gamma}$ has no critical point in $(0,2)\times(0,1)$.

Summarizing, we conclude that the maximum value of $F_{\mu,\gamma}$ is attained on the boundary of R. Taking into account (2.18) and (2.20), as in [18, p. 10] the following cases hold. For $|\gamma| \geq 2/(1+\mu)$ the maximum value of $F_{\mu,\gamma}$ is attained in a corner of R, namely,

$$\max_{(x,y)\in R} F_{\mu,\gamma}(x,y) = F_{\mu,\gamma}(2,1) = 2(1+2\mu)|\gamma|.$$

For $\sqrt{2/(1+\mu)} \leq |\gamma| < 2/(1+\mu)$ the maximum value of $F_{\mu,\gamma}$ is attained in $(x_{\mu,\gamma},1)$, i.e.,

$$\max_{(x,y)\in R} F_{\mu,\gamma}(x,y) = F_{\mu,\gamma}(x_{\mu,\gamma},1).$$

For $0 < |\gamma| < \sqrt{2/(1+\mu)}$ we compare all values (2.17), and by (2.19) and (2.21), the values $F_{\mu,\gamma}(x_{\mu,\gamma},1)$ and $F_{\mu,\gamma}(2,y_{\mu,\gamma})$ and we show that the maximum value of $F_{\mu,\gamma}$ is attained in $(x_{\mu,\gamma},1)$. The key of the computation is to show that

$$F_{\mu,\gamma}(x_{\mu,\gamma},1) \geq F_{\mu,\gamma}(2,y_{\mu,\gamma}).$$

Putting (2.19) and (2.21) into above, we get the inequality which is identical as the inequality (2.42) of [18] and further the proof follows exactly as in [18, pp. 10–13] (similar method of proof with all details can be found in [21]). Going back to (2.16) with $\mu = (\alpha+\beta)/2$ we conclude that the following inequality holds:

$$\max_{f\in\mathcal{C}(g_{\alpha,\beta})} \Phi_\lambda(f) \leq \begin{cases} \dfrac{1}{6}\left|\alpha^2+\beta^2+\dfrac{1}{2}(\alpha+\beta)^2\gamma\right| + \dfrac{1}{3}(1+\alpha+\beta)|\gamma|, & |\gamma| \geq \dfrac{4}{2+\alpha+\beta}, \\ \dfrac{1}{6}\left|\alpha^2+\beta^2+(\alpha+\beta)^2\gamma\right| + \dfrac{(\alpha+\beta)^2\gamma^2}{12(2-|\gamma|)} + \dfrac{2}{3}, & |\gamma| \leq \dfrac{4}{2+\alpha+\beta}. \end{cases}$$

Setting $\gamma = 2-3\lambda$ the above result yields the inequality (2.5).

Now we discuss the sharpness of the result. Let $\alpha,\beta \in [0,1]$, $(\alpha,\beta) \neq (0,0)$. Let $\lambda \in [\tau_1(\alpha,\beta), 2/3]$. Then we consider the second inequality in (2.5) which after simple computing is

$$\max_{f\in\mathcal{C}(g_{\alpha,\beta})} \Phi_\lambda(f) \leq \frac{(\alpha+\beta)^2}{9\lambda} + \frac{2-\alpha\beta}{3}. \tag{2.22}$$

Let $t_{\alpha,\beta,\lambda} := (\alpha+\beta)(1/(3\lambda)-1/2)$. Since $\tau_1(\alpha,\beta) \leq \lambda \leq 2/3$, so $0 \leq t_{\alpha,\beta,\lambda} \leq 1$. Thus in view of (2.4), $p_{t_{\alpha,\beta,\lambda},0} \in \mathcal{P}$ with $c_1 = 2t_{\alpha,\beta,\lambda}$ and $c_2 = 2$. Setting $\delta := 0$ and $p := p_{t_{\alpha,\beta,\lambda},0}$ into (2.13) we get the function $f_{\alpha,\beta,t_{\alpha,\beta,\lambda}}$ given by (2.6) for which, by (2.14),

$$\begin{aligned} a_2 &= t_{\alpha,\beta,\lambda} + (\alpha+\beta)/2 = (\alpha+\beta)/(3\lambda), \\ a_3 &= (2+2(\alpha+\beta)t_{\alpha,\beta,\lambda} + \alpha^2+\alpha\beta+\beta^2)/3 \\ &= 2(\alpha+\beta)^2/(9\lambda) + (2-\alpha\beta)/3, \end{aligned} \tag{2.23}$$

and which makes the equality in (2.22).

Let now $\lambda \in \mathbf{R} \setminus (\tau_1(\alpha,\beta), \lambda(\alpha,\beta))$. Since $\lambda(\alpha,\beta) \geq \tau_2(\alpha,\beta)$, we consider the first inequality in (2.5) which, taking also into account that $\tau_1(\alpha,\beta) \leq 2/3$, after computing, is

$$\max_{f\in\mathcal{C}(g_{\alpha,\beta})} \Phi_\lambda(f) \leq \left|\frac{2}{3} + \frac{2}{3}(\alpha+\beta) + \frac{1}{3}(\alpha^2+\alpha\beta+\beta^2) - \frac{1}{4}(2+\alpha+\beta)^2\lambda\right|. \tag{2.24}$$

Setting $\delta := 0$ and $p := L$ into (2.13) we get the function $f_{\alpha,\beta,1}$ given by (2.6) with $t_{\alpha,\beta,\lambda} := 1$ and with the coefficients a_2 and a_3 given by (2.23), which makes the equality in (2.24). In particular, the function $f_{\alpha,\beta,1}$ is one of the form (2.7)–(2.10).

Let $\alpha = \beta := 0$. For $\lambda \in [\tau_1(0,0), \tau_2(0,0)] = [0, 4/3]$ the inequality (2.5) reduces to

$$\max_{f\in\mathcal{C}(g_{0,0})} \Phi_\lambda(f) = \max_{f\in\mathcal{P}'} \Phi_\lambda(f) \leq \frac{2}{3}.$$

Setting $\delta := 0$ and by (2.4), $p := p_{0,0}$ into (2.13) we get the function (2.11) with $a_2 = 0$ and $a_3 = 2/3$, which makes the equality above. For $\lambda \in \mathbf{R} \setminus (0, 4/3)$ the inequality (2.5) reduces to

$$\max_{f \in \mathcal{C}(g_{0,0})} \Phi_\lambda(f) = \max_{f \in \mathcal{P}'} \Phi_\lambda(f) \le \left| \frac{2}{3} - \lambda \right|.$$

Setting $\delta := 0$ and by (2.4), $p := L$ into (2.13) we get the function (2.12) with $a_2 = 1$ and $a_3 = 2/3$, which makes the equality above. □

Remark 2.3. *Let* $\beta := \alpha \in (0, 1]$. *Then by* (1.5) *we have*

$$\lambda(\alpha, \alpha) = \frac{2}{3} + \frac{2}{3} \max\left\{ \frac{1}{2}, \frac{1}{1+\alpha} \right\} = \tau_2(\alpha),$$

so (2.5) *with sharpness reduces to* (1.4) *(Theorem 2.4 of [18]). Let* $\beta := 0$ *and* $\alpha \in (0, 1]$. *Then*

$$\lambda(\alpha, 0) = \frac{2}{3} + \frac{2}{3} \max\left\{ 1, \frac{2}{2+\alpha} \right\} = \frac{4}{3},$$

so (2.5) *with sharpness reduces to* (1.6) *(Theorem 2.4 of [21]).*

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O PEWNEJ NIERÓWNOŚCI DLA WSPÓŁCZYNNIKÓW W PODKLASIE FUNKCJI PRAWIE WYPUKŁYCH

S t r e s z c z e n i e

Dla $\alpha, \beta \in [0,1]$ niech $g_{\alpha,\beta}(z) := z//((1-\alpha z)(1-\beta z)),\ z \in \mathbf{D} := \{z \in \mathbf{C} : |z| < 1\}$. Funkcja analityczna unormowana $f : \mathbf{D} \to \mathbf{C}$ nazywna jest *prawie wypukłą względem funkcji* $g_{\alpha,\beta}$, jeśli dla pewnego $\delta \in (-\pi//2, \pi//2)$ zachodzi nierówność

$$\mathrm{Re}\left\{ \mathrm{e}^{\mathrm{i}\delta} \frac{zf'(z)}{g_{\alpha,\beta}(z)} \right\} > 0, \quad z \in \mathbf{D}.$$

Dla klasy $\mathcal{C}(g_{\alpha,\beta})$ funkcji prawie wypukłych względem funkcji $g_{\alpha,\beta}$ badany jest problem Fekete-Szegö-Guluzina.

Słowa kluczowe: nierówności współczynnikowe, funkcje prawie wypukłe, problem Fekete-Szegö-Goluzina