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REDUCED INCIDENCE ALGEBRAS DESCRIPTION OF COBWEB POSETS AND KoDAGs

Summary
    The notion of reduced incidence algebra of an arbitrary cobweb poset is delivered. The incidence algebra combinatorial characterization for an arbitrary positive integers valued sequence is given.

1. Cobweb posets

The family of the so called cobweb posets Π has been invented by A. K. Kwaśniewski few years ago (for references see: [6, 7]). These structures are such a generalization of the Fibonacci tree growth that allows joint combinatorial interpretation for all of them under the admissibility condition (see [8, 9]).

Let $\{F_n\}_{n \geq 0}$ be a positive integers valued sequence with $F_0 = 1$ (with $F_0 = 0$ being exceptional as in case of Fibonacci numbers). Any sequence satisfying this property uniquely designates cobweb poset defined as follows.

For $s \in \mathbf{N_0} = \mathbf{N} \cup \{\mathbf{0}\}$ let us to define levels of Π:

$$\Phi_s = \{\langle j, s \rangle, \ 1 \leq j \leq F_s\},$$

(in case of $F_0 = 0$ level Φ_0 corresponds to the empty root $\{\emptyset\}$). Then

Definition 1. *Corresponding cobweb poset is an infinite partially ordered set* $\Pi = (V, \leq)$*, where*

$$V = \bigcup_{0 \leq s} \Phi_s$$

are the elements (vertices) of Π *and the partial order relation* $\leq$ *on* V *for* $x = \langle s, t \rangle, y = \langle u, v \rangle$ *being elements of cobweb poset* Π *defined by formula*

$$(x \leq_P y) \Longleftrightarrow [(t < v) \vee (t = v \wedge s = u)].$$

Obviously any cobweb poset can be represented, via its Hasse diagram, as infinite directed graph $\Pi = (V, E)$, where set V of its vertices is defined as above and

$$E = \{(\langle j, p \rangle, \langle q, (p+1) \rangle)\} \ \cup \ \{(\langle 1, 0 \rangle, \langle 1, 1 \rangle)\},$$

where $1 \leq j \leq F_p$ and $1 \leq q \leq F_{(p+1)}$ stay for set of (directed) edges.

The Kwaśniewski cobweb posets under consideration represented by graphs are examples of orderable directed acyclic graphs (oDAG) which we start to call from now in brief: KoDAGs. These are structures of universal importance for the whole of mathematics and computer sciences in general (quotation from [8, 9]):

> For any given natural numbers valued sequence the graded (layered) cobweb posets' DAGs are equivalently representations of a chain of binary relations. Every relation of the cobweb poset chain is biunivocally represented by the uniquely designated **complete** bipartite digraph-a digraph which is a di-biclique designated by the very given sequence. The cobweb poset is then to be identified with a chain of di-bicliques i.e. by definition–a chain of complete bipartite one direction digraphs. Any chain of relations is therefore obtainable from the cobweb poset chain of complete relations via deleting arcs (arrows) in di-bicliques.

According to the definition above arbitrary cobweb poset $\Pi = (V, \leq)$ is a graded poset (ranked poset) and for $s \in \mathbf{N_0}$:

$$x \in \Phi_s \quad \longrightarrow \quad r(x) = s,$$

where $r : \Pi \to \mathbf{N_0}$ is a rank function on Π.

Let us then define Kwaśniewski finite cobweb sub-posets as follows

Definition 2. *Let* $P_n = (V_n, \leq)$, $(n \geq 0)$, *for* $V_n = \bigcup_{0 \leq s \leq n} \Phi_s$ *and* $\leq$ *being the induced partial order relation on* Π.

It is easy to see that P_n is ranked poset with rank function r as above. P_n has a unique minimal element $0 = \langle 1, 0 \rangle$ (with $r(0) = 0$). Moreover Π and all P_n-s are locally finite, i.e. for any pair $x, y \in \Pi$, the segment $[x, y] = \{z \in \Pi : x \leq z \leq y\}$ is finite.

Let us recall that one defines the incidence algebra of a locally finite partially ordered set P as follows (see [10–12]):

$$I(P) = I(P, R) = \{f : P \times P \longrightarrow R; \quad f(x, y) = 0 \quad unless \quad x \leq y\}.$$

The sum of two such functions f and g and multiplication by scalar are defined as usual. The product $h = f * g$ is defined as follows:

$$h(x,y) = (f*g)(x,y) = \sum_{z \in \mathbf{P}:\ x \leq z \leq y} f(x,z) \cdot g(z,y).$$

It is immediately verified that this is an associative algebra (with an identity element $\delta(x,y)$, the Kronecker delta), over any associative ring R.

In [5] the incidence algebra of an arbitrary cobweb poset Π (or its subposets P_n) uniquely designated by the natural numbers valued sequence $\{F_n\}_{n \geq 0}$, was considered by the present author. The explicit formulas for some typical elements of incidence algebra $I(\Pi)$ of Π where delivered there.

So for x, y being some arbitrary elements of Π such that $x = \langle s,t \rangle$, $y = \langle u,v \rangle$, $(s, u \in \mathbf{N}, t, v \in \mathbf{N_0})$, $1 \leq s \leq F_t$ and $1 \leq u \leq F_v$ one has:

(1) ζ function of Π being a characteristic function of partial order in Π

$$\text{(1)} \qquad \zeta(x,y) = \zeta(\langle s,t \rangle, \langle u,v \rangle) = \delta(s,u)\delta(t,v) + \sum_{k=1}^{\infty} \delta(t+k, v),$$

one can also verify, that ζ^k enumerates all multichains of length k,

(2) Möbius function of Π being a inverse of ζ

$$\text{(2)} \quad \mu(x,y) = \delta(t,v)\delta(s,u) - \delta(t+1,v) + \sum_{k=2}^{\infty} \delta(t+k,v)(-1)^k \prod_{i=t+1}^{v-1}(F_i - 1),$$

(3) function $\zeta^2 = \zeta * \zeta$ counting the number of elements in the segment $[x,y]$

$$\text{(3)} \qquad\qquad \zeta^2(x,y) = card\,[x,y] \quad = \quad \left(\sum_{i=t+1}^{v-1} F_i \right) + 2,$$

(4) function η

$$\text{(4)} \qquad\qquad \eta(x,y) = \sum_{k=1}^{\infty} \delta(t+k,v) = \begin{cases} 1 & t < v \\ 0 & w\ p.p. \end{cases},$$

(5) function $\eta^k(x,y)$, $(k \in \mathbf{N})$ counting the number of chains of length k, (with $(k+1)$ elements) from x to y

$$\text{(5)} \qquad \begin{aligned} \eta^k(x,y) \quad &= \sum_{x < z_1 < z_2 < \ldots < z_{k-1} < y} 1 \\ &= \sum_{t < i_1 < i_2 < \ldots < i_{k-1} < v} F_{i_1} F_{i_2} \ldots F_{i_{k-1}}, \end{aligned}$$

(6) function $\mathcal{C}$

$$(6) \qquad \mathcal{C}(\langle s,t \rangle, \langle u,v \rangle) = \delta(t,v)\delta(s,u) - \sum_{k=1}^{\infty} \delta(t+k,v),$$

such that its inverse function $\mathcal{C}^{-1}(x,y)$ counts the number of all chains from x to y'

(7) function χ

$$(7) \qquad \chi(x,y) = \delta(t+1,v),$$

(8) function $\chi^k(x,y)$, $(k \in \mathbf{N})$ counting the number of maximal chains of length k, (with $(k+1)$ elements) from x to y

$$(8) \qquad \chi^k(x,y) = \sum_{x \lessdot z_1 \lessdot \ldots \lessdot z_{k-1} \lessdot y} 1 = \delta(t+k,v)F_{t+1}F_{t+2}\ldots F_{v-1}.$$

(9) function $\mathcal{M}$

$$(9) \qquad \mathcal{M}(\langle s,t \rangle, \langle u,v \rangle) = \delta(t,v)\delta(s,u) - \delta(t+1,v),$$

such that its inverse function $\mathcal{M}^{-1}$ counts the number of all maximal chains from x to y.

In this paper the notion of the standard reduced incidence algebra [1, 11, 12] of an arbitrary cobweb poset Π will be delivered. As we shall see, it enables us for example to facilitate the formulas presented above. The results presented below stay true when considering finite subposets P_n defined above.

2. The Standard Reduced Incidence Algebra of an arbitrary cobweb poset

Let $S(\Pi)$ be the set of all segments in Π and let $\sim$ be the equivalence relation $\sim \subseteq S(\Pi) \times S(\Pi)$.

Let us recall that $\sim$ is compatible ([1]), i.e. it satisfies the following condition: if f and g belong to the incidence algebra $I(P)$ and $f(x,y) = f(u,v)$ as well as $g(x,y) = g(u,v)$ for all pairs of segments such that $[x,y] \sim [u,v]$, then $(f * g)(x,y) = (f * g)(u,v)$.

The equivalence classes of segments of Π relative to $\sim$ are called types. The set of all functions defined on types (i.e. all functions taking the same value on equivalent segments) forms an associative algebra with identity. One calls it the reduced incidence algebra $R(\Pi, \sim)$ (modulo of equivalence relation $\sim$). Let us note that $R(\Pi, \sim)$ is isomorphic to a subalgebra of $I(\Pi)$, ([1]).

Now let be $\sim$ defined as follows

$$(10) \qquad [x,y] \sim [u,v] \iff segments \; [x,y], [u,v] \; are \; isomorphic.$$

One can show that it is order compatible. Then one calls $R(\Pi, \sim) = R(\Pi)$ the standard reduced incidence algebra of Π. Also from the definitions of $\sim$ and partial order on Π one infers that

$$[x, y] \sim [u, v] \Longleftrightarrow [r(x) = r(u) \wedge r(y) = r(v)].$$

So let T be the set of types of relation $\sim$ defined above. Then

$$T = \{(k, n) : k, n \in \mathbf{N_0}\}$$

and for $k \leq n$

(11) $$(k, n) = \{[x, y] \in S(\Pi) : r(x) = k \wedge r(y) = n\},$$

or equivalently

$$(k, n) = \{[x, y] \in S(\Pi) : x \in \Phi_k \wedge y \in \Phi_n\}.$$

Also let $(k, n) = \emptyset$ for $k > n$.

Definition 3. *Let* $[x, y] \in \alpha_{k, n}$. *For* $l \in \mathbb{N}_0$ *one can define the incidence coefficients in* $R(\Pi)$ *as follows*

(12) $$\left\langle \begin{array}{c} k, n \\ l \end{array} \right\rangle = |\{z \in [x, y] : [x, z] \in (k, l) \wedge [z, y] \in (l, n)\}|.$$

The following formula holds.

Proposition 1.

(13) $$\left\langle \begin{array}{c} k, n \\ l \end{array} \right\rangle = \left\{ \begin{array}{ll} F_l & k \leq l \leq n \\ 0 & otherwise \end{array} \right. .$$

Now one can define the product $*$ in $R(\Pi)$ as follows.

Proposition 2. *Let* $[x, y] \in (k, n)$, $(k \leq n)$ *and* $f, g \in R(\Pi)$. *Then*

(14) $$(f * g)(k, n) = \sum_{l \geq 0} F_l f(k, l) g(l, n)$$

with the assumption that $(k, n) = \emptyset$, *for* $l < k$ *or* $l > n$.

$$(f * g)(k, \, n) \quad = \quad (f * g)(x, y) = \sum_{x \le z \le y} f(x, z) g(z, y)$$

$$= \sum_{k \le l \le n} \sum_{\{z : [x, z] \in (k, \, l), \, [z, y] \in (l, \, n)\}} f(x, z) g(z, y)$$

Proof.

$$= \sum_{k \le l \le n} F_l f(k, \, l) g(l, \, n)$$

$$= \sum_{l \ge 0} F_l f(k, \, l) g(l, \, n).$$

$\square$

So we have proved

Theorem 1. *Let $F = \{F_n\}_{n \ge 0}$ z $F_0 = 1$, be an arbitrary natural numbers valued sequence with $F_0 = 1$(with $F_0 = 0$ being exceptional). The the numbers F_n ($n \ge 0$) are the incidence coefficients in the standard reduced algebra $R(\Pi)$ of cobweb poset Π uniquely designated by the sequence $F = \{F_n\}_{n \ge 0}$.*

One can also show the following

Theorem 2. *Let $f \in R(\Pi)$. Then for $x, y \in \Pi$ such that $[x, y] \in (k, \, n)$ the value $f(x, y)$ depends on $r(x) = k$ and $r(y) = n$) only, i.e.*

(15) $$f(x, y) = f(k, n) = f(r(x), r(y)).$$

From the definition of partial order on Π one can also infer that for $x, y \in \Pi$ satisfying $r(x) = r(y)$ and for $f \in I(\Pi)$, one has

$$f(x, y) = \delta(x, y).$$

It is known that all elements of $I(\Pi)$ mentioned above, i.e. functions: ζ, μ, ζ^2, ζ^k, η, η^k, $\mathcal{C}$, $\mathcal{C}^{-1}$, χ, χ^k, $\mathcal{M}$, $\mathcal{M}^{-1}$ are the elements of an arbitrary reduced incidence algebra $R(\Pi, \sim)$, (i.e. modulo an arbitrary order compatible equivalence relation $\sim$ on $S(\Pi)$). Then the next results follow immediately from this fact and above theorems.

Corollary 1. *Let $(k, n) \in T$. Then:*

(16) $$\zeta(k, n) = \begin{cases} 1 & k \le n \\ 0 & k > n \end{cases};$$

(17) $$\zeta^2(k, n) = \begin{cases} \left(\displaystyle\sum_{i=k}^{n-1} F_i \right) + 2 & k \le n \\ 0 & k > n \end{cases};$$

$$(18) \qquad \eta(k,n) = \delta_{k<n} = \begin{cases} 1 & k < n \\ 0 & k \geq n \end{cases} ;$$

$$(19) \qquad \eta^2(k,n) = \begin{cases} \sum_{i=k+1}^{n-1} F_i & k \leq n \\ 0 & k > n \end{cases} ;$$

$$(20) \qquad \eta^s(k,n) = \begin{cases} \sum_{k<i_1<...<i_{s-1}<n} F_{i_1} F_{i_2}...F_{i_{k-1}} & k \leq n \\ 0 & k > n \end{cases} ;$$

$$(21) \qquad \mathcal{C}(k,n) = \begin{cases} 1 & k = n \\ -1 & k < n \\ 0 & k > n \end{cases} ;$$

$$(22) \qquad \chi(k,n) = \delta(k+1,n);$$

$$(23) \qquad \chi^s(k,n) = \delta(k+s,n) \cdot F_{k+1} F_{k+2}...F_{n-1};$$

$$(24) \qquad \mu(k,n) = \begin{cases} (-1)^{n-k} \prod_{i=k+1}^{n-1} (F_i - 1) & k \leq n \\ 0 & k > n \end{cases} .$$

Corollary 2. *The standard reduced incidence algebra $R(\Pi)$ is the maximally reduced incidence algebra $\overline{R}(\Pi)$, i.e. the smallest reduced incidence algebra on Π. Equivalently the equivalence relation defined by (10) is the maximal element in the lattice of all order compatible equivalence relations on $S(\Pi)$.*

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OPIS COBWEB POSETÓW W JĘZYKU ZREDUKOWANYCH ALGEBR INCYDENCJI

S t r e s z c z e n i e

W pracy opisano strukturę zredukowanej algebry incydencji dowolnego cobweb posetu. Na tej podstawie podano kombinatoryczną interpretację dowolnego ciągu liczbowego o wartościach ze zbioru liczb naturalnych.

Słowa kluczowe: zbiór częściowo uporządkowany (poset), cobweb poset, algebra incydencji zbioru częściowo uporządkowanego lokalnie skończonego, zredukowana algebra incydencji