https://doi.org/10.26485/0459-6854/2018/68.2/10 PL ISSN 0459-6854

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2018 Vol. LXVIII

Recherches sur les déformations no. 2

pp. 95–108

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THE SCHWARZ TYPE INEQUALITY FOR HARMONIC FUNCTIONS OF THE UNIT DISC SATISFYING A SECTORIAL CONDITION

Summary

Let T_1, T_2 and T_3 be closed arcs contained in the unit circle $\mathbb{T}$ with the same length $2\pi/3$ and covering $\mathbb{T}$. In the paper [3] D. Partyka and J. Zając obtained the sharp estimation of the module $|F(z)|$ for $z \in \mathbb{D}$ where $\mathbb{D}$ is the unit disc and F is a complex-valued harmonic function of $\mathbb{D}$ into itself satisfying the following sectorial condition: For each $k \in \{1, 2, 3\}$ and for almost every $z \in T_k$ the radial limit of the function F at the point z belongs to the angular sector determined by the convex hull spanned by the origin and arc T_k. In this article a more general situation is considered where the three arcs are replaced by a finite collection $T_1, T_2, \ldots, T_n$ of closed arcs contained in $\mathbb{T}$ with positive length, total length 2π and covering $\mathbb{T}$.

Keywords and phrases: harmonic functions, Harmonic mappings, Poisson integral, Schwarz Lemma

1. Introduction

Throughout the paper we always assume that all topological notions and operations are understood in the complex plane $\mathrm{E}(\mathbb{C}) := (\mathbb{C}, \rho_e)$, where ρ_e is the standard euclidean metric. We will use the notations $\mathrm{cl}(A)$ and $\mathrm{fr}(A)$ for the closure and boundary of a set $A \subset \mathbb{C}$ in $\mathrm{E}(\mathbb{C})$, respectively. By $\mathrm{Har}(\Omega)$ we denote the class of all complex-valued harmonic functions in a domain Ω, i.e., the class of all twice continuously differentiable functions F in Ω satisfying the Laplace equation

$$\frac{\partial^2 F(z)}{\partial x^2} + \frac{\partial^2 F(z)}{\partial y^2} = 0, \quad z = x + \mathrm{i}y \in \Omega.$$

The sets $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ are the unit disc and unit circle, respectively. The standard measure of a Lebesgue measurable set $A \subset \mathbb{T}$ will be denoted by $|A|_1$. In particular, if A is an arc then $|A|_1$ means its length. Set $\mathbb{Z}_{p,q} := \{k \in \mathbb{Z} : p \leq k \leq q\}$ for any $p, q \in \mathbb{Z}$.

Definition 1.1. For every $n \in \mathbb{N}$ a sequence $\mathbb{Z}_{1,n} \ni k \mapsto T_k \subset \mathbb{T}$ is said to be a *partition of the unit circle* provided T_k is a closed arc of length $|T_k|_1 > 0$ for $k \in \mathbb{Z}_{1,n}$ as well as

$$\bigcup_{k=1}^{n} T_k = \mathbb{T} \quad \text{and} \quad \sum_{k=1}^{n} |T_k|_1 = 2\pi. \tag{1.1}$$

For any function $F : \mathbb{D} \to \mathbb{C}$ and $z \in \mathbb{T}$ we define the set $F^{**}(z)$ of all $w \in \mathbb{C}$ such that there exists a sequence $\mathbb{N} \ni n \mapsto r_n \in [0; 1)$ satisfying the equalities

$$\lim_{n \to +\infty} r_n = 1 \quad \text{and} \quad \lim_{n \to +\infty} F(r_n z) = w.$$

Definition 1.2. By the *sectorial boundary normalization given by a partition* $\mathbb{Z}_{1,n} \ni k \mapsto T_k \subset \mathbb{T}$ *of the unit circle* we mean the class $\mathcal{N}(T_1, T_2, \ldots, T_n)$ of all functions $F : \mathbb{D} \to \mathbb{D}$ such that for every $k \in \mathbb{Z}_{1,n}$ and almost every (a.e. in abbr.) $z \in T_k$,

$$F^{**}(z) \subset D_k := \{ru : 0 \leq r \leq 1\,,\, u \in T_k\} = \operatorname{conv}(T_k \cup \{0\}). \tag{1.2}$$

Given $n \in \mathbb{N}$ and a partition $\mathbb{Z}_{1,n} \ni k \mapsto T_k \subset \mathbb{T}$ of the unit circle we will study the Schwarz type inequality for the class

$$\mathcal{F} := \operatorname{Har}(\mathbb{D}) \cap \mathcal{N}(T_1, T_2, \ldots, T_n).$$

If $n \leq 2$ then we have a trivial sharp estimation $|F(z)| \leq 1$ for $F \in \mathcal{F}$ and $z \in \mathbb{D}$, where the equality is attained for a constant function. Therefore, from now on we always assume that $n \geq 3$.

In Section 2 we prove a few useful properties of the class $\mathcal{F}$. Most essential here is Theorem 2.3. We use it to show in Section 3 Theorem 3.1, which is our main result. Then we apply the last theorem in specific cases; cf. Examples 3.4 and 3.5. In particular, we derive the estimation (3.13), obtained by D. Partyka and J. Zając in [3, Corollary 2.2]. Thus the estimation (3.1), valid for an arbitrary partition of $\mathbb{T}$, generalizes the one (3.13), which holds only in the case where $n = 3$ and the arcs T_1, T_2 and T_3 have the same length. Note that the estimation (3.12) is a directional improvement of the radial one (3.13). In Example 3.5 we study a general case of an arbitrary partition of the unit circle. As a result, we derive reasonable estimations (3.23) and (3.24), which depend on the largest length among the ones $|T_k|_1$ for $k \in \mathbb{Z}_{1,n}$.

2. Auxiliary results

Let P$[f]$ stand for the Poisson integral of an integrable function $f : \mathbb{T} \to \mathbb{C}$, i.e., P$[f] : \mathbb{D} \to \mathbb{C}$ is the function given by the following formula

$$\mathrm{P}[f](z) := \frac{1}{2\pi} \int_{\mathbb{T}} f(u) \frac{1-|z|^2}{|u-z|^2} |\mathrm{d}u| = \frac{1}{2\pi} \int_{\mathbb{T}} f(u) \operatorname{Re} \frac{u+z}{u-z} |\mathrm{d}u|, \; z \in \mathbb{D}. \tag{2.1}$$

The Poisson integral provides the unique solution to the Dirichlet problem in the unit disc $\mathbb{D}$ provided that the boundary function f is continuous. It means that P$[f]$ is a harmonic function in $\mathbb{D}$, which has a continuous extension to the closed disc cl$(\mathbb{D})$ and its boundary values function coincides with f. For any function $F : \mathbb{D} \to \mathbb{C}$ we define the radial limit function of F by the formula

$$\mathbb{T} \ni z \mapsto F^*(z) := \begin{cases} \lim\limits_{r \to 1^-} F(rz), & \text{if the limit exists,} \\ 0, & \text{otherwise.} \end{cases}$$

Since a real-valued harmonic and bounded function in $\mathbb{D}$ has the radial limit for a.e. point of $\mathbb{T}$ (see e.g. [2, Cor. 1, Sect. 1.2]), it follows that $F^* = (\operatorname{Re} F)^* + \mathrm{i}(\operatorname{Im} F)^*$ almost everywhere on $\mathbb{T}$ provided $F \in \operatorname{Har}(\mathbb{D})$ is bounded in $\mathbb{D}$. Therefore,

$$F^{**}(z) = \{F^*(z)\} \quad \text{for every } F \in \mathcal{F} \text{ and a.e. } z \in \mathbb{T}. \tag{2.2}$$

In particular, for each function $F : \mathbb{D} \to \mathbb{D}$, $F \in \mathcal{F}$ if and only if $F \in \operatorname{Har}(\mathbb{D})$ and $F^*(z) \in D_k$ for $k \in \mathbb{Z}_{1,n}$ and a.e. $z \in T_k$. From the property (2.2) it follows that for each $F \in \mathcal{F}$ the sequence $\mathbb{N} \ni m \mapsto f_m$, where

$$\mathbb{T} \ni u \mapsto f_m(u) := F((1 - \tfrac{1}{m})u), \quad m \in \mathbb{N},$$

is convergent to F^* almost everywhere on $\mathbb{T}$. Then applying the dominated convergence theorem we see that for every $z \in \mathbb{D}$,

$$F((1 - \tfrac{1}{m})z) = \mathrm{P}[f_m](z) \to \mathrm{P}[F^*](z) \quad \text{as } m \to +\infty,$$

which yields

$$F = \mathrm{P}[F^*], \quad F \in \mathcal{F}. \tag{2.3}$$

Let χ_I be the characteristic function of a set $I \in \mathbb{T}$, i.e., $\chi_I(t) := 1$ for $t \in I$ and $\chi_I(t) := 0$ for $t \in \mathbb{T} \setminus I$.

Lemma 2.1. *For all $F \in \mathcal{F}$ and $z \in \mathbb{D}$ there exists a sequence $\mathbb{Z}_{1,n} \ni k \mapsto c_k \in D_k$ such that the following equality holds*

$$F(z) = \sum_{k=1}^{n} c_k \, \mathrm{P}[\chi_{T_k}](z). \tag{2.4}$$

Proof. Fix $F \in \mathcal{F}$ and $z \in \mathbb{D}$. Since $|T_k|_1 > 0$ for $k \in \mathbb{Z}_{1,n}$, it follows that

$$0 < p_k := \mathrm{P}[\chi_{T_k}](z) < 1, \quad k \in \mathbb{Z}_{1,n}. \tag{2.5}$$

By (1.2) each sector D_k, $k \in \mathbb{Z}_{1,n}$, is closed and convex. Moreover, from (1.2) and (2.2) we see that $F^*(z) \in D_k$ for $k \in \mathbb{Z}_{1,n}$ and a.e. $z \in T_k$. Then applying the integral mean value theorem for complex-valued functions we deduce from (2.5) that

$$c_k := P\Big[\frac{1}{p_k} \cdot F^* \cdot \chi_{T_k}\Big](z) \in D_k, \quad k \in \mathbb{Z}_{1,n}.$$

Hence and by (2.3),

$$\begin{aligned} F(z) = \mathrm{P}[F^*](z) = \mathrm{P}\Big[\sum_{k=1}^{n} F^* \cdot \chi_{T_k}\Big](z) &= \sum_{k=1}^{n} \mathrm{P}\Big[F^* \cdot \chi_{T_k}\Big](z) \\ &= \sum_{k=1}^{n} p_k \, \mathrm{P}\Big[\frac{1}{p_k} \cdot F^* \cdot \chi_{T_k}\Big](z) = \sum_{k=1}^{n} p_k c_k, \end{aligned}$$

which implies the equality (2.4). □

Lemma 2.2. *For every sequence $\mathbb{Z}_{1,n} \ni k \mapsto c_k \in D_k$,*

$$F := \sum_{k=1}^{n} c_k \, \mathrm{P}[\chi_{T_k}] \in \mathcal{F}. \tag{2.6}$$

Proof. Given a sequence $\mathbb{Z}_{1,n} \ni k \mapsto c_k \in D_k$ consider the function F defined by the formula (2.6). Since $\mathrm{P}[\chi_{T_k}] \in \mathrm{Har}(\mathbb{D})$ for $k \in \mathbb{Z}_{1,n}$, we see that $F \in \mathrm{Har}(\mathbb{D})$. Furthermore, for each $z \in \mathbb{D}$,

$$\sum_{k=1}^{n} \mathrm{P}[\chi_{T_k}](z) = \mathrm{P}\Big[\sum_{k=1}^{n} \chi_{T_k}\Big](z) = \mathrm{P}[\chi_{\mathbb{T}}](z) = 1,$$

whence

$$|F(z)| \leq \sum_{k=1}^{n} |c_k| \, \mathrm{P}[\chi_{T_k}](z) \leq \sum_{k=1}^{n} \mathrm{P}[\chi_{T_k}](z) = 1.$$

By the definition of the function F we have

$$F^*(z) = \sum_{k=1}^{n} c_k \chi_{T_k}(z), \quad z \in \mathbb{T} \setminus E, \tag{2.7}$$

where E is the set of all $u \in \mathbb{T}$ such that u is an endpoint of a certain arc among the arcs T_k for $k \in \mathbb{Z}_{1,n}$.

Assume that $|F(z_0)| = 1$ for some $z_0 \in \mathbb{D}$. By the maximum modulus principle for complex-valued harmonic functions (cf. [1, Corollary 1.11, p. 8]) there exists $w \in \mathbb{T}$ such that $F(z) = w$ for $z \in \mathbb{D}$, and so $F^*(z) = w$ for $z \in \mathbb{T}$. By (2.7), $F^*(z) = c_k$ for $k \in \mathbb{Z}_{1,n}$ and $z \in T_k \setminus E$. Therefore $w = c_k \in D_k$ for $k \in \mathbb{Z}_{1,n}$, and so $w \in D_1 \cap D_2 \cap D_3 = \{0\}$. Hence $w = 0$, which contradicts the equality $|w| = 1$. Thus $F(z) < 1$ for $z \in \mathbb{D}$, and so $F : \mathbb{D} \to \mathbb{D}$. Furthermore, from (2.7) it follows that for all $k \in \mathbb{Z}_{1,n}$ and $z \in T_k \setminus E$, $F^*(z) = c_k \in D_k$. Thus $F \in \mathcal{N}(T_1, T_2, \ldots, T_n)$, which implies (2.6). □

Theorem 2.3. *For every compact set $K \subset \mathbb{D}$ there exist a sequence $\mathbb{Z}_{1,n} \ni k \mapsto c_k \in D_k$ and $z_K \in \mathrm{fr}(K)$ such that*

$$F_K := \sum_{k=1}^{n} c_k \,\mathrm{P}[\chi_{T_k}] \in \mathcal{F} \tag{2.8}$$

and

$$|F(z)| \le |F_K(z_K)| = \Big|\sum_{k=1}^{n} c_k \,\mathrm{P}[\chi_{T_k}](z_K)\Big|, \quad F \in \mathcal{F}\,,\, z \in K. \tag{2.9}$$

In particular,

$$\max(\{|F(z)| : F \in \mathcal{F}\,,\, z \in K\}) = |F_K(z_K)|. \tag{2.10}$$

Proof. Fix a compact set $K \subset \mathbb{D}$. Since $F(K) \subset F(\mathbb{D}) \subset \mathbb{D}$ for $F \in \mathcal{F}$,

$$M_K := \sup(\{|F(z)| : F \in \mathcal{F}\,,\, z \in K\}) \le 1. \tag{2.11}$$

Hence, there exist sequences $\mathbb{N} \ni m \mapsto F_m \in \mathcal{F}$ and $\mathbb{N} \ni m \mapsto z_m \in K$ such that

$$\lim_{m\to+\infty} |F_m(z_m)| = M_K. \tag{2.12}$$

From Lemma 2.1 it follows that for each $m \in \mathbb{N}$ there exists a sequence $\mathbb{Z}_{1,n} \ni k \mapsto c_{m,k} \in D_k$ such that

$$F_m(z_m) = \sum_{k=1}^{n} c_{m,k} \,\mathrm{P}[\chi_{T_k}](z_m). \tag{2.13}$$

Since the set D_k is compact for $k \in \mathbb{Z}_{1,n}$ we see, using the standard technique of choosing a convergent subsequence from a sequence in a compact set, that there exists an increasing sequence $\mathbb{N} \ni l \mapsto m_l \in \mathbb{N}$, a sequence $\mathbb{Z}_{1,n} \ni k \mapsto c_k \in D_k$ and $z'_K \in K$ such that

$$c_{m_l,k} \to c_k \quad \text{as } l \to +\infty \quad \text{for } k \in \mathbb{Z}_{1,n} \tag{2.14}$$

and

$$z_{m_l} \to z'_K \quad \text{as } l \to +\infty. \tag{2.15}$$

By Lemma 2.2, the property (2.8) holds. From (2.13) we conclude that for every $m \in \mathbb{N}$,

$$\begin{aligned} |F_K(z_m) - F_m(z_m)| &= \Big|\sum_{k=1}^{n} c_k \,\mathrm{P}[\chi_{T_k}](z_m) - \sum_{k=1}^{n} c_{m,k} \,\mathrm{P}[\chi_{T_k}](z_m)\Big| \\ &\le \sum_{k=1}^{n} |c_k - c_{m,k}| \,\mathrm{P}[\chi_{T_k}](z_m) \\ &\le \sum_{k=1}^{n} |c_k - c_{m,k}|, \end{aligned}$$

which together with (2.14) leads to

$$\lim_{l\to+\infty} |F_K(z_{m_l}) - F_{m_l}(z_{m_l})| = 0. \tag{2.16}$$

Since $|c_k| \leq 1$ for $k \in \mathbb{Z}_{1,n}$, it follows that

$$\begin{aligned}|F_K(z'_K) - F_K(z_m)| &\leq \Big|\sum_{k=1}^{n} c_k \,\mathrm{P}[\chi_{T_k}](z'_K) - \sum_{k=1}^{n} c_k \,\mathrm{P}[\chi_{T_k}](z_m)\Big| \\ &\leq \sum_{k=1}^{n} |c_k| \cdot |\,\mathrm{P}[\chi_{T_k}](z'_K) - \mathrm{P}[\chi_{T_k}](z_m)| \\ &\leq \sum_{k=1}^{n} |\,\mathrm{P}[\chi_{T_k}](z'_K) - \mathrm{P}[\chi_{T_k}](z_m)|, \quad m \in \mathbb{N}.\end{aligned}$$

This together with (2.15) yields

$$\lim_{l\to+\infty} |F_K(z'_K) - F_K(z_{m_l})| = 0. \tag{2.17}$$

Since for every $l \in \mathbb{N}$,

$$|F_K(z'_K) - F_{m_l}(z_{m_l})| \leq |F_K(z'_K) - F_K(z_{m_l})| + |F_K(z_{m_l}) - F_{m_l}(z_{m_l})|,$$

we deduce from (2.17) and (2.16) that

$$\lim_{l\to+\infty} |F_{m_l}(z_{m_l})| = |F_K(z'_K)|.$$

Hence and by (2.12), $|F_K(z'_K)| = M_K$. Since $F_K \in \mathrm{Har}(\mathbb{D})$, the maximum modulus principle for complex-valued harmonic function (cf. [1, Corollary 1.11, p. 8]) implies that there exists $z_K \in \mathrm{fr}(K)$ such that $|F_K(z)| \leq |F_K(z_K)|$ for $z \in K$. In particular, $M_K = |F_K(z'_K)| \leq |F_K(z_K)|$. On the other hand, by (2.8) and (2.11), $|F_K(z_K)| \leq M_K$. Eventually, $|F_K(z_K)| = M_K$. This implies (2.10), and thereby, the inequality (2.9) holds, which is the desired conclusion. □

3. Estimations

As an application of Theorem 2.3 we shall prove the following result.

Theorem 3.1. *For every $z \in \mathbb{D}$ the following inequality holds*

$$|F(z)| \leq 1 - (n - S)p(z), \quad F \in \mathcal{F}, \tag{3.1}$$

where

$$S := \sup\Big(\Big\{\mathrm{Re}\Big(\overline{u}\sum_{k=1}^{n} v_k\Big) : u \in \mathbb{T}\,,\, \mathbb{Z}_{1,n} \ni k \mapsto v_k \in D_k\Big\}\Big) \tag{3.2}$$

and

$$p(z) := \min(\{\mathrm{P}[\chi_{T_k}](z) : k \in \mathbb{Z}_{1,n}\}). \tag{3.3}$$

Proof. It is clear that $K := \{z\}$ is a compact set for a given $z \in \mathbb{D}$. By Theorem 2.3 there exists a sequence $\mathbb{Z}_{1,n} \ni k \mapsto c_k \in D_k$ such that

$$F_K := \sum_{k=1}^{n} c_k \,\mathrm{P}[\chi_{T_k}] \in \mathcal{F}$$

and

$$|F(z)| \le |F_K(z)|, \quad F \in \mathcal{F}. \tag{3.4}$$

Setting $u := F_K(z)/|F_K(z)|$ if $F_K(z) \neq 0$ and $u := 1$ if $F_K(z) = 0$, we see that $u \in \mathbb{T}$ and $F_K(z) = u|F_K(z)|$. Hence

$$|F_K(z)| = \overline{u}F_K(z) = \mathrm{Re}(\overline{u}F_K(z)) = \mathrm{Re}\Big(\overline{u}\sum_{k=1}^{n} c_k p_k\Big) = \sum_{k=1}^{n} \mathrm{Re}(\overline{u}c_k)p_k, \tag{3.5}$$

where $p_k := \mathrm{P}[\chi_{T_k}](z)$ for $k \in \mathbb{Z}_{1,n}$. Since

$$\sum_{k=1}^{n} p_k = 1 \quad \text{and} \quad \mathrm{Re}(\overline{u}c_k) \le M := \max(\{\mathrm{Re}(\overline{u}c_l) : l \in \mathbb{Z}_{1,n}\}) \le 1, \quad k \in \mathbb{Z}_{1,n},$$

we deduce from the formula (3.3) that

$$\begin{aligned}
\sum_{k=1}^{n} \mathrm{Re}(\overline{u}c_k)p_k &= \sum_{k=1}^{n}(\mathrm{Re}(\overline{u}c_k) - M + M)p_k \\
&= M\sum_{k=1}^{n} p_k + \sum_{k=1}^{n}(\mathrm{Re}(\overline{u}c_k) - M)p_k \\
&\le M\sum_{k=1}^{n} p_k + \sum_{k=1}^{n}(\mathrm{Re}(\overline{u}c_k) - M)p(z) \\
&= M\sum_{k=1}^{n}(p_k - p(z)) + p(z)\sum_{k=1}^{n}\mathrm{Re}(\overline{u}c_k) \\
&\le \sum_{k=1}^{n}(p_k - p(z)) + p(z)\sum_{k=1}^{n}\mathrm{Re}(\overline{u}c_k) \\
&= 1 - np(z) + p(z)\sum_{k=1}^{n}\mathrm{Re}(\overline{u}c_k).
\end{aligned}$$

This together with (3.5) and (3.2) yields

$$\begin{aligned}
|F_K(z)| &\le 1 - np(z) + p(z)\sum_{k=1}^{n}\mathrm{Re}(\overline{u}c_k) \\
&\le 1 - np(z) + p(z)S \\
&= 1 - (n - S)p(z).
\end{aligned}$$

Hence and by (3.4) we obtain the estimation (3.1), which proves the theorem. □

The estimation (3.1) is useful provided we can estimate $p(z)$ from below and S from above. The first task is easy and depends on the following quantity

$$\delta := \frac{1}{2}\min(\{|T_k|_1 : k \in \mathbb{Z}_{1,n}\}). \tag{3.6}$$

Lemma 3.2. *For every $\alpha \in (0;\pi/2]$ the following estimation holds*

$$\mathrm{P}[\chi_{I_\alpha}](z) \geq \mathrm{P}[\chi_{I_\alpha}](|z|) = \frac{2}{\pi}\arctan\left(\frac{\sin(\alpha)}{|z|+\cos(\alpha)}\right) - \frac{\alpha}{\pi}, \quad z \in \mathbb{D}, \tag{3.7}$$

where $I_\alpha := \{\mathrm{e}^{\mathrm{i}t} : |t-\pi| \leq \alpha\}$.

Proof. Given $\alpha \in (0;\pi/2]$ we see that $e_1 := \mathrm{e}^{\mathrm{i}(\pi-\alpha)} = -\mathrm{e}^{-\mathrm{i}\alpha}$ and $e_2 := \mathrm{e}^{\mathrm{i}(\pi+\alpha)} = -\mathrm{e}^{\mathrm{i}\alpha}$ are the endpoints of the arc I_α. Let $z \in \mathbb{D}$ be arbitrarily fixed. Since $I_\alpha \subset \Omega_z := \mathbb{C} \setminus \{z+t : t>0\}$, the function $\Omega_z \ni \zeta \mapsto \log(z-\zeta)$ is holomorphic and

$$\frac{\mathrm{d}}{\mathrm{d}t}\log(z-\mathrm{e}^{\mathrm{i}t}) = \frac{\mathrm{i}\mathrm{e}^{\mathrm{i}t}}{\mathrm{e}^{\mathrm{i}t}-z}, \quad t \in [\pi-\alpha;\pi+\alpha].$$

Here we understand the function log as the inverse of the function $\exp_{|\Omega}$, where $\Omega := \{\zeta \in \mathbb{C} : |\operatorname{Im}\zeta| < \pi\}$. By (2.1) we have

$$\begin{aligned}
\mathrm{P}[\chi_{I_\alpha}](z) &= \frac{1}{2\pi}\int_{\mathbb{T}} \chi_{I_\alpha}(u)\operatorname{Re}\frac{u+z}{u-z}|\mathrm{d}u| \\
&= \frac{1}{2\pi}\int_{\pi-\alpha}^{\pi+\alpha} \operatorname{Re}\frac{\mathrm{e}^{\mathrm{i}t}+z}{\mathrm{e}^{\mathrm{i}t}-z}\mathrm{d}t \\
&= \frac{1}{2\pi}\int_{\pi-\alpha}^{\pi+\alpha} \operatorname{Re}\Big(\frac{2\mathrm{e}^{\mathrm{i}t}}{\mathrm{e}^{\mathrm{i}t}-z} - 1\Big)\mathrm{d}t \\
&= \frac{1}{\pi}\int_{\pi-\alpha}^{\pi+\alpha} \operatorname{Im}\Big(\frac{\mathrm{i}\mathrm{e}^{\mathrm{i}t}}{\mathrm{e}^{\mathrm{i}t}-z}\Big)\mathrm{d}t - \frac{\alpha}{\pi} \\
&= \frac{1}{\pi}\int_{\pi-\alpha}^{\pi+\alpha} \operatorname{Im}\frac{\mathrm{d}}{\mathrm{d}t}\log(z-\mathrm{e}^{\mathrm{i}t})\mathrm{d}t - \frac{\alpha}{\pi} \\
&= \frac{1}{\pi}\operatorname{Im}\big[\log(z-e_2) - \log(z-e_1)\big] - \frac{\alpha}{\pi}.
\end{aligned}$$

Therefore, for an arbitrarily fixed $r \in [0;1)$,

$$\mathrm{P}[\chi_{I_\alpha}](r\mathrm{e}^{\mathrm{i}\theta}) = \frac{1}{\pi}\operatorname{Im}\big[\log(r\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{\mathrm{i}\alpha}) - \log(r\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{-\mathrm{i}\alpha})\big] - \frac{\alpha}{\pi}, \quad \theta \in \mathbb{R}. \tag{3.8}$$

Consequently,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\theta}\mathrm{P}[\chi_{I_\alpha}](r\mathrm{e}^{\mathrm{i}\theta}) &= \frac{1}{\pi}\operatorname{Im}\Big[\frac{\mathrm{i}r\mathrm{e}^{\mathrm{i}\theta}}{r\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{\mathrm{i}\alpha}} - \frac{\mathrm{i}r\mathrm{e}^{\mathrm{i}\theta}}{r\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{-\mathrm{i}\alpha}}\Big] \\
&= \frac{r}{\pi}\operatorname{Im}\Big[\frac{\mathrm{i}\mathrm{e}^{\mathrm{i}\theta}(-\mathrm{e}^{\mathrm{i}\alpha}+\mathrm{e}^{-\mathrm{i}\alpha})}{(r\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{\mathrm{i}\alpha})(r\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{-\mathrm{i}\alpha})}\Big] \\
&= \frac{2r\sin(\alpha)}{\pi}\,\frac{\operatorname{Im}[\mathrm{e}^{\mathrm{i}\theta}(r\mathrm{e}^{-\mathrm{i}\theta}+\mathrm{e}^{-\mathrm{i}\alpha})(r\mathrm{e}^{-\mathrm{i}\theta}+\mathrm{e}^{\mathrm{i}\alpha})]}{|r\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{\mathrm{i}\alpha}|^2|r\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{-\mathrm{i}\alpha}|^2} \\
&= \frac{2r\sin(\alpha)}{\pi}\,\frac{\operatorname{Im}[r^2\mathrm{e}^{-\mathrm{i}\theta}+r\mathrm{e}^{-\mathrm{i}\alpha}+r\mathrm{e}^{\mathrm{i}\alpha}+\mathrm{e}^{\mathrm{i}\theta}]}{|r\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{\mathrm{i}\alpha}|^2|r\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{-\mathrm{i}\alpha}|^2} \\
&= \frac{2r(1-r^2)\sin(\alpha)\sin(\theta)}{\pi|r\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{\mathrm{i}\alpha}|^2|r\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{-\mathrm{i}\alpha}|^2}, \quad \theta \in \mathbb{R}.
\end{aligned}$$

Combining this with (3.8) we derive the estimation (3.7), which proves the lemma. □

Corollary 3.3. *The following estimation holds*

$$p(z) \geq \mathrm{P}[\chi_{I_\delta}](|z|) = \frac{2}{\pi} \arctan\left(\frac{\sin(\delta)}{|z| + \cos(\delta)}\right) - \frac{\delta}{\pi}, \quad z \in \mathbb{D}, \tag{3.9}$$

where $p(z)$ and δ are defined by the formulas (3.3) *and* (3.6), *respectively.*

Proof. Let $\mathbb{Z}_{1,n} \ni k \mapsto a_k \in \mathbb{T}$ be the sequence of midpoints of the partition $\mathbb{Z}_{1,n} \ni k \mapsto T_k \subset \mathbb{T}$, i.e.,

$$T_k := \{a_k \mathrm{e}^{\mathrm{i}t} : |t| \leq \alpha_k\}, \tag{3.10}$$

where $\alpha_k := \frac{1}{2}|T_k|_1$ for $k \in \mathbb{Z}_{1,n}$. Hence and by (3.6) we obtain $I_\delta \subset I_{\alpha_k}$ for $k \in \mathbb{Z}_{1,n}$, where $I_\alpha := \{\mathrm{e}^{\mathrm{i}t} : |t - \pi| \leq \alpha\}$ for $\alpha \in (0;\pi]$. Then applying the formula (2.1) we see that for an arbitrarily fixed $z \in \mathbb{D}$,

$$\mathrm{P}[\chi_{I_{\alpha_k}}](|z|) = \mathrm{P}[\chi_{I_\delta}](|z|) + \mathrm{P}[\chi_{I_{\alpha_k} \setminus I_\delta}](|z|) \geq \mathrm{P}[\chi_{I_\delta}](|z|), \quad k \in \mathbb{Z}_{1,n}.$$

Therefore

$$\min(\{\mathrm{P}[\chi_{I_{\alpha_k}}](|z|) : k \in \mathbb{Z}_{1,n}\}) = \mathrm{P}[\chi_{I_\delta}](|z|), \tag{3.11}$$

because $\delta = \alpha_{k'}$ for some $k' \in \mathbb{Z}_{1,n}$. Fix $k \in \mathbb{Z}_{1,n}$. Using the rotation mapping $\mathbb{C} \ni \zeta \mapsto \varphi(\zeta) := -a_k^{-1}\zeta$ we have $\varphi(T_k) = I_{\alpha_k}$. Then integrating by substitution we deduce from the formula (2.1) that

$$\mathrm{P}[\chi_{T_k}](z) = \mathrm{P}[\chi_{\varphi(T_k)}](\varphi(z)) = \mathrm{P}[\chi_{I_{\alpha_k}}](\varphi(z)).$$

On the other hand, by Lemma 3.2,

$$\mathrm{P}[\chi_{I_{\alpha_k}}](\varphi(z)) \geq \mathrm{P}[\chi_{I_{\alpha_k}}](|\varphi(z)|) = \mathrm{P}[\chi_{I_{\alpha_k}}](|z|).$$

Thus

$$\mathrm{P}[\chi_{T_k}](z) \geq \mathrm{P}[\chi_{I_{\alpha_k}}](|z|), \quad k \in \mathbb{Z}_{1,n}.$$

Combining this with (3.3) and (3.11) we derive the estimation (3.9), which completes the proof. □

A more difficult problem is to estimate from above the quantity S given by the formula (3.2). It will be studied elsewhere. Now we present two examples.

Example 3.4. Suppose that $\mathbb{Z}_{1,3} \ni k \mapsto T_k \subset \mathbb{T}$ is a partition of $\mathbb{T}$ such that $|T_1|_1 = |T_2|_1 = |T_3|_1$. As in the proof of [3, Theorem 2.1] we can show that $S \leq 2$. Hence and by Theorem 3.1 we obtain

$$|F(z)| \leq 1 - p(z) = 1 - \min(\{\mathrm{P}[\chi_{T_k}](z) : k \in \mathbb{Z}_{1,3}\}), \quad F \in \mathcal{F}, z \in \mathbb{D}. \tag{3.12}$$

Corollary 3.3 now implies the estimation

$$|F(z)| \leq \frac{4}{3} - \frac{2}{\pi} \arctan\left(\frac{\sqrt{3}}{1 + 2|z|}\right), \quad F \in \mathcal{F}, z \in \mathbb{D}; \tag{3.13}$$

cf. [3, Corollary 2.2]. Therefore, the estimation (3.12) is a directional type enhancement of the radial one (3.13) for the class $\mathcal{F}$.

Example 3.5. Suppose that $\mathbb{Z}_{1,n} \ni k \mapsto T_k \subset \mathbb{T}$ is a partition of $\mathbb{T}$ such that

$$\Delta := \max(\{|T_k|_1 : k \in \mathbb{Z}_{1,n}\}) \leq \frac{\pi}{2}. \tag{3.14}$$

Then

$$N := \mathrm{Ent}\Big(\frac{\pi}{2\Delta}\Big) \geq 1. \tag{3.15}$$

Fix $u \in \mathbb{T}$ and a sequence $\mathbb{Z}_{1,n} \ni k \mapsto v_k \in D_k$. There exist a bijective function σ of the set $\mathbb{Z}_{1,n}$ onto itself and an increasing sequence $\mathbb{Z}_{1,n} \ni k \mapsto \alpha_k \in \mathbb{R}$ such that $\alpha_n = 2\pi + \alpha_0$, $u \in T_{\sigma(1)}$ and

$$T_{\sigma(k)} = \{\mathrm{e}^{\mathrm{i}t} : \alpha_{k-1} \leq t \leq \alpha_k\}, \quad k \in \mathbb{Z}_{1,n}.$$

Hence there exist $\theta \in [\alpha_0; \alpha_1]$ and a sequence $\mathbb{Z}_{1,n} \ni k \mapsto (r_k, \theta_k) \in [0;1] \times \mathbb{R}$ such that $u = \mathrm{e}^{\mathrm{i}\theta}$, $v_k = r_k \mathrm{e}^{\mathrm{i}\theta_k}$ for $k \in \mathbb{Z}_{1,n}$ and

$$\alpha_{k-1} \leq \theta_{\sigma(k)} \leq \alpha_k, \quad k \in \mathbb{Z}_{1,n}. \tag{3.16}$$

Since for each $k \in \mathbb{Z}_{1,n}$,

$$\mathrm{Re}(\overline{u} v_k) = \mathrm{Re}\Big(r_k \mathrm{e}^{\mathrm{i}\theta_k} \mathrm{e}^{-\mathrm{i}\theta}\Big) = \mathrm{Re}\Big(r_k \mathrm{e}^{\mathrm{i}(\theta_k - \theta)}\Big) = r_k \cos(\theta_k - \theta),$$

we conclude that

$$\mathrm{Re}(\overline{u} v_k) \leq \max(\{0\,, \cos(\theta_k - \theta)\}), \quad k \in \mathbb{Z}_{1,n}. \tag{3.17}$$

From (3.14) it follows that

$$\alpha_j - \alpha_i = \sum_{l=i+1}^{j} (\alpha_l - \alpha_{l-1}) \leq (j-i)\Delta, \quad i,j \in \mathbb{Z}_{0,n}\,,\ i < j. \tag{3.18}$$

Setting

$$p := \min(\{k \in \mathbb{Z}_{1,n} : \alpha_k \geq \frac{\pi}{2} + \theta\}) \quad \text{and} \quad q := \max(\{k \in \mathbb{Z}_{1,n} : \alpha_k < \frac{3\pi}{2} + \theta\})$$

we conclude from (3.15) and (3.18) that

$$N\Delta \leq \frac{\pi}{2} \leq \alpha_p - \theta \leq \alpha_p - \alpha_0 \leq p\Delta$$

as well as

$$N\Delta \leq \frac{\pi}{2} = \alpha_q + \frac{\pi}{2} - \alpha_q < 2\pi + \theta - \alpha_q \leq \alpha_n - \alpha_q + \alpha_1 - \alpha_0 \leq (n - q + 1)\Delta.$$

Therefore $N \leq p$ and $q + N \leq n$. Given $k \in \mathbb{Z}_{1,n}$ the following four cases can appear. If $p + 1 - N \leq k \leq p$ then by (3.16) and (3.18),

$$\frac{\pi}{2} + \theta - \theta_{\sigma(k)} \leq \alpha_p - \alpha_{k-1} \leq (p + 1 - k)\Delta \leq N\Delta \leq \frac{\pi}{2}$$

as well as

$$\frac{\pi}{2} + \theta - \theta_{\sigma(k)} > \alpha_{p-1} - \alpha_p \geq -\Delta \geq -\frac{\pi}{2},$$

which gives

$$\cos(\theta_{\sigma(k)} - \theta) = \sin(\pi/2 + \theta - \theta_{\sigma(k)}) \le \sin((p+1-k)\Delta).$$

Hence and by (3.17) we obtain

$$\mathrm{Re}(\overline{u} v_{\sigma(k)}) \le \sin((p+1-k)\Delta), \quad k \in \mathbb{Z}_{p+1-N,p}\,. \tag{3.19}$$

If $p+1 \le k \le q$ then by (3.16),

$$\frac{\pi}{2} + \theta \le \alpha_{k-1} \le \theta_{\sigma(k)} \le \alpha_k < \frac{3\pi}{2} + \theta,$$

and so $\cos(\theta_{\sigma(k)} - \theta) \le 0$. This together with (3.17) leads to

$$\mathrm{Re}(\overline{u} v_{\sigma(k)}) \le 0, \quad k \in \mathbb{Z}_{p+1,q}\,. \tag{3.20}$$

If $q+1 \le k \le q+N$ then by (3.16) and (3.18),

$$\theta_{\sigma(k)} - \frac{3\pi}{2} - \theta \le \alpha_k - \frac{3\pi}{2} - \theta < \alpha_k - \alpha_q \le (k-q)\Delta \le N\Delta \le \frac{\pi}{2}$$

as well as

$$\theta_{\sigma(k)} - \frac{3\pi}{2} - \theta \ge \alpha_q - \alpha_{q+1} \ge -\Delta \ge -\frac{\pi}{2},$$

and consequently,

$$\cos(\theta_{\sigma(k)} - \theta) = \sin(\theta_{\sigma(k)} - 3\pi/2 - \theta) \le \sin((k-q)\Delta).$$

Hence and by (3.17) we obtain

$$\mathrm{Re}(\overline{u} v_{\sigma(k)}) \le \sin((k-q)\Delta), \quad k \in \mathbb{Z}_{q+1,q+N}. \tag{3.21}$$

If $1 \le k \le p-N$ or $q+N+1 \le k \le n$, then clearly $\mathrm{Re}(\overline{u} v_{\sigma(k)}) \le 1$. Combining this with (3.19), (3.20) and (3.21) we see that

$$\begin{aligned} \sum_{k=1}^{n} \mathrm{Re}(\overline{u} v_{\sigma(k)}) &\le \sum_{k=p+1-N}^{p} \sin((p+1-k)\Delta) + \sum_{k=q+1}^{q+N} \sin((k-q)\Delta) \\ &\quad + (p-N) + (n-q-N) \\ &= 2\sum_{k=1}^{N} \sin(k\Delta) + n - 2N - (q-p). \end{aligned} \tag{3.22}$$

Since $\pi < \alpha_{q+1} - \alpha_{p-1} \le (q-p+2)\Delta$, we deduce from (3.15) that $2N \le q-p+1$. Combining this with (3.22) we get

$$\begin{aligned} \sum_{k=1}^{n} \mathrm{Re}(\overline{u} v_{\sigma(k)}) &\le 2\sum_{k=1}^{N} \sin(k\Delta) + n - 2N - (2N-1) \\ &= n + 1 - 4N + 2\frac{\sin\left(\frac{(N+1)\Delta}{2}\right)\sin\left(\frac{N\Delta}{2}\right)}{\sin\left(\frac{\Delta}{2}\right)}. \end{aligned}$$

Hence and by (3.2),

$$S \le n + 1 - 4N + 2\frac{\sin\left(\frac{(N+1)\Delta}{2}\right)\sin\left(\frac{N\Delta}{2}\right)}{\sin\left(\frac{\Delta}{2}\right)}.$$

Theorem 3.1 now shows that

$$|F(z)| \le 1 - \left(4N - 1 - 2\frac{\sin\left(\frac{(N+1)\Delta}{2}\right)\sin\left(\frac{N\Delta}{2}\right)}{\sin\left(\frac{\Delta}{2}\right)}\right)p(z), \quad F \in \mathcal{F}, z \in \mathbb{D}, \tag{3.23}$$

where N and $p(z)$ are defined by (3.15) and (3.3), respectively. Applying now Corollary 3.3 we derive from (3.23) the following estimation of radial type

$$|F(z)| \le 1 - \left(4N - 1 - 2\frac{\sin\left(\frac{(N+1)\Delta}{2}\right)\sin\left(\frac{N\Delta}{2}\right)}{\sin\left(\frac{\Delta}{2}\right)}\right)\mathrm{P}[\chi_{I_\delta}](|z|),$$
$$F \in \mathcal{F}, z \in \mathbb{D}, \tag{3.24}$$

where δ is given by the formula (3.6).

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Presented by Zbigniew Jakubowski at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on April 16, 2018.

NIERÓWNOŚCI TYPU SCHWARZA DLA FUNKCJI HARMONICZNYCH W KOLE JEDNOSTKOWYM SPEŁNIAJĄCYCH PEWIEN WARUNEK SEKTOROWY

S t r e s z c z e n i e

Niech T_1, T_2 i T_3 będą łukami domkniętymi, zawartymi w okręgu jednostkowym $\mathbb{T}$, o tej samej długości $2\pi/3$ i pokrywającymi $\mathbb{T}$. W pracy [3] D. Partyka and J. Zając otrzymali dokładne oszacowanie modułu $|F(z)|$ dla $z \in \mathbb{D}$, gdzie $\mathbb{D}$ jest kołem jednostkowym, zaś F jest funkcją harmoniczną o wartościach zespolonych koła $\mathbb{D}$ w siebie, spełniających następujący warunek sektorowy: dla każdego $k \in \{1, 2, 3\}$ i prawie każdego $z \in T_k$ granica radialna funkcji F w punkcie z należy do sektora kątowego będącego otoczką wypukłą zbioru $\{0\} \cup T_k$. W tym artykule rozważamy ogólniejszy przypadek, gdzie trzy łuki są zastąpione przez skończony układ łuków domkniętych $T_1, T_2, \ldots, T_n$ zawartych w $\mathbb{T}$, o dodatniej długości, całkowitej długości 2π i pokrywających $\mathbb{T}$.

Słowa kluczowe: całka Poissona, funkcje harmoniczne, lemat Schwarza, odwzorowania harmoniczne