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Dedicated to the memory of
Professor Yurii B. Zelinskii

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METHOD FOR GENERATING OF ARCHIMEDEAN TRIANGULAR CONNECTIVE OPERATORS

Summary

Archimedean triangular operators form the basis of the tools for development, analysis and designing of fuzzy systems. They are generated by simple, monotone, single-valued and continuous functions. The basic sets of logical connectives of fuzzy systems are analyzed. It is shown that they are divided into conditional and algebraic ones. The well-known methods for generating of Archimedean triangular operators are described. Limitation of the functional characteristics of such generated operators is shown. To expand them, a method for generating has been created, which makes it possible to build new parameterized operators. Examples of constructing both new operators and those that generalize the already known ones are given. The tendency of the change of the characteristic hyper surface, which is build by new operator, is revealed.

Keywords and phrases: T-norms, S-norms, T-operators, connective generators

1. Introduction

In 1965 L. Zadeh published an article [1], which initiated the theory of fuzzy sets. Three operations of the classic set theory - union, intersection and complement - play the leading role in it. These operations are now often referred to as t-norm (T), t-conorm (S), and negation (N). L. Zadeh suggested using the minimum function for the operation of fuzzy sets intersection, the maximum function for the operation of the union, and the negation function for the complement operation. Nowadays they became basic not only in fuzzy logic, but also in fuzzy control systems, fuzzy pattern recognition and various industrial fuzzy applications [2]. They are called min-max operators. These operators were assigned to the class of conditional ones [3–5]. However, not only they began to be used. Dubois and Prade [6] showed that another operator on the basis of product (product operator) in some cases is better than min-operator. These and similar operators in [7] were classified into the class of algebraic operators. Then different types of triangular norms, introduced by K. Menger [8] became the acquisition of the theory of fuzzy sets, which resulted in the rapid development of fuzzy logic. It was the development of the theory of triangular norms that provided the basis for the creation of mathematical means of formulating statements for decision-making. Today, fuzzy systems have been created, which are the basis of management and decision-making in many branches of industry, manufacturing, science and medicine [9]. In these, the above operators carry out the functions of fuzzy logical connectives of two or more variables implemented by the corresponding triangular norms, in particular t-norm $T(x,y)$, t-conorm $S(x,y)$, and negation $N(x)$. Therefore, the construction of these operators with new functional characteristics is an actual task.

Based on this, the purpose of the paper is to construct a method for generation of logical connectives operators for fuzzy systems with new properties of its characteristics. These operators would be parameterized and generalize some of the existing logical connectives. For this in the beginning we shall briefly consider the basic properties of the main operators of logical connectives - triangular T-norms and S-norms, then represent a new method for generating, which is characterized by expanded functional properties of generated connectives and give some examples of them.

2. Basic properties of logical connective operators

L. Zadeh [1] and other [12] describe the operators T, S, and N as those from which the classical sets of logical connectives operators are constructed. There are a number of axioms that they must satisfy. Let's describe them. So, for the T-norm

$T : [0,\ 1] \times [0,\ 1] \to [0,\ 1]$ and for $x, y, z \in [0,\ 1]$ the next conditions must be satisfied:

$T(0,\ 0) = T(0,\ 1) = T(1,\ 0) = 0$ and $T(1,\ 1) = 1$ (boundary conditions),

$T(x,y) = T(y,x)$ (commutativity),

$T(x,y) \le T(x,z)$, if $y \le z$ (monotonicity),

$T(T(x,y),\ z) = T(x,\ T(y,z))$ (associativity).

It is also necessary to comply with such requirements:

$T(x,1) = x$.

It is said that the triangular norm T is an Archimedean under the conditions:

T is continuous,

$T(x,\, x) < x$ for arbitrary $x \in [0,\, 1]$ (condition of strictness).

The corresponding axioms must also satisfy the S-norms

$S : [0,\, 1] \times [0,\, 1] \to [0,\, 1]$ and for $x, y, z \in [0,\, 1]$ next conditions must be satisfied:

$S(0, 0) = 0$ and $S(0,\, 1) = S(1,\, 0) = S(1,\, 1) = 1$ (boundary conditions),

$S(x, y) = S(y, x)$ (commutativity),

$S(x, y) \le S(x, z)$, if $y \le z$ (monotonicity),

$S(S(x, y), z) = S(x, S(y, z))$ (associativity).

It is also necessary to comply with such requirements:

$S(x, 0) = x$.

It is said that the triangular conorm S is Archimedean under the conditions:

S is continuous,

$S(x, x) > x$ for arbitrary $x \in [0, 1]$ (condition of strictness).

Proceeding from the variety of existing triangular norms [12], they are divided into a class of conditional operators and a class of algebraic operators [7]. Let's briefly describe some of them.

3. Selected conditional and algebraic logical connective operators

The class of conditional operators includes the above-mentioned operators proposed by L. Zadeh [1]:

$$N(x) = 1 - x, \tag{1}$$

$$T_Z(x, y) = \min(x, y), \tag{2}$$

$$S_Z(x, y) = \max(x, y). \tag{3}$$

These three operators form a triplet. Among them, let's pay attention to the negation operator $N(x)$, because it is an integral part of the triplet of other types. Therefore, presenting them further, the operator $N(x)$ will not be indicated. Herewith, we note that operators $T(x)$ and $S(x)$ are interconnected by De-Morgan's rule

$$S(x, y) = 1 - T(1 - x,\ 1 - y). \tag{4}$$

Also known are such conditional operators [2, 3], like Lukasiewicz

$$T_{£}(x, y) = \max(x + y - 1,\ 0), \tag{5}$$

$$S_{£}(x, y) = \min(x + y,\ 1) \tag{6}$$

Yager [13]

$$T_Y(x, y) = \max\left(1 - \left((1 - x)^\alpha + (1 - y)^\alpha\right)^{1/\alpha},\ 0\right), \tag{7}$$

$$S_Y(x, y) = \min\left((x^\alpha + y^\alpha)^{1/\alpha},\ 1\right) \tag{8}$$

where $\alpha > 0$, Fodor [14]

$$T_F(x,y) = \begin{cases} \min(x,y), & if\ x+y > 1, \\ 0, & if\ x+y \le 1, \end{cases} \tag{9}$$

$$S_F(x,y) = \begin{cases} \max(x,y), & if\ x+y < 1, \\ 1, & if\ x+y \ge 1, \end{cases} \tag{10}$$

Dubois and Prade [15]

$$T_{D,P}(x,y) = \frac{xy}{\max(x,y,\ \alpha)}, \tag{11}$$

where $\alpha > 0$

$$S_{D,P}(x,y) = \frac{x+y-xy-\min(x,y,1-\alpha)}{\min(1-x,1-y,\ \alpha)} \tag{12}$$

To the class of algebraic operators [7] belong ones, which are based on such strict Archimedean T-norms as algebraic:

$$T_a(x,y) = xy, \tag{13}$$

$$S_a(x,y) = x+y-xy, \tag{14}$$

Einstein [12]

$$T_E(x,y) = \frac{xy}{2-(x+y-xy)}, \tag{15}$$

$$S_E(x,y) = \frac{x+y}{1+xy}, \tag{16}$$

Hamacher [16]

$$T_H(x,y) = \frac{xy}{x+y-xy}, \tag{17}$$

$$S_H(x,y) = \frac{x+y-2xy}{1-xy}, \tag{18}$$

Frank [17]

$$T_F(x,y) = \log_p(1+\frac{(p^x-1)(p^y-1)}{p-1}), \tag{19}$$

$$S_F(x,y) = 1-\log_p(1+\frac{(p^{1-x}-1)(p^{1-y}-1)}{p-1}), \tag{20}$$

Dombi [4]

$$T_D(x,y) = \frac{1}{1+((\frac{1-x}{x})^\alpha+(\frac{1-y}{y})^\alpha)^{1/\alpha}}, \tag{21}$$

$$S_D(x,y) = 1-\frac{1}{1+((\frac{x}{1-x})^\alpha+(\frac{y}{1-y})^\alpha)^{1/\alpha}}. \tag{22}$$

where $\alpha > 0$.

However, the way of generation of analytical expressions of the above-described operators remains unclear. Therefore, we will briefly describe the generators with the help of which the above-described operators were received. Herewith, let us just consider only generators of strict Archimedean triangular norms as a means of constructing of logical connectives.

4. Known methods for generating of logical connective operators

The above operators of strict Archimedean triangular norms are obtained through the use of a generating method, which is described by the expression [11, 12]

$$h_C(x,y) = f^{-1}(f(x) + f(y)), \tag{23}$$

where $x, y \in [0,\ 1]$, $f(x) \in [0, \infty)$, $\lim\limits_{x\to 0} f(x) = \infty$, $f(1) = 0$, generator $f(\cdot)$ is continuous function and $f^{-1}(\cdot)$ is a function inverse to $f(\cdot)$. This method is directly related to the Cauchy functional equation

$$f(x + y) = f(x) + f(y). \tag{24}$$

The method of a bit different kind than the expression (23) which is associated with the solution of the functional equation [18]

$$f(x + y) = f(x) + f(y) + f(x)f(y) \tag{25}$$

was considered in [7]. By analogy with (23), it will take the form

$$h_A(x,y) = f^{-1}[f(x) + f(y) + f(x)f(y)], \tag{26}$$

where $x, y \in [0, 1)$; $\lim\limits_{x\to 0} f(x) = \infty$, $f(1) = 0$. However, such a generating method (26) has no possibility of direct parameterization of the received operators. Therefore, in [19] its generalization is described by introducing a parameterizing coefficient $p > 0$:

$$h_V(x,y) = f^{-1}[f(x) + f(y) + pf(x)f(y)]. \tag{27}$$

However, the above-described generating method of logical connective operators have limited the ability to influence on the properties of their characteristics. Therefore, we will further describe the generating method to obtain the flexible logical connectives with the new properties and give their examples.

5. New generating method of logical connective operators

Developing methods for logical connectives generating [11, 12, 18, 20, 21] we propose different from (27) expression

$$h(x,y) = f^{-1}\{\frac{1}{p}\ln[e^{pf(x)} + e^{pf(y)} - 1]\}. \tag{28}$$

where $x, y \in [0,\ 1]$, $p > 0$, $f(x) \in [0,\ \infty)$, $\lim\limits_{x\to 0} f(x) = \infty$, $f(1) = 0$, and $f^{-1}(\cdot)$ is the inverse to the function-generator$f$. Lets prove that the generating method (28) satisfies the requirements of such axioms:

1. Commutativity

$$h(x,y) = f^{-1}\{\frac{1}{p}\ln[e^{pf(x)} + e^{pf(y)} - 1]\} = f^{-1}\{\frac{1}{p}\ln[e^{pf(y)} + e^{pf(x)} - 1]\} = h(y,x).$$

2. Associativity

$$h(h(x,y),z) = f^{-1}\{\frac{1}{p}\ln[\exp(f(f^{-1}(\frac{1}{p}\ln(e^{pf(x)} + e^{pf(y)} - 1)))) + e^{pf(z)} - 1]\} =$$

$$= f^{-1}\{\frac{1}{p}\ln[e^{pf(x)} + e^{pf(y)} + e^{pf(z)} - 2]\} =$$

$$= f^{-1}\{\frac{1}{p}\ln[e^{pf(x)} + \exp(pf(f^{-1}(\frac{1}{p}\ln(e^{pf(y)} + e^{pf(z)} - 1)))) - 1]\} =$$

$$= f^{-1}\{\frac{1}{p}\ln[e^{pf(x)} + \exp(pf(h(y,z))) - 1]\} = h(x, h(y,z)).$$

3. Monotonicity. Let $y \le z$. Then, taking into account that f is a monotonically decreasing function and $p > 0$, we can write the inequality

$$e^{pf(y)} \ge e^{pf(z)}.$$

Adding to both sides of inequality $e^{pf(x)}$ we will have

$$e^{pf(x)} + e^{pf(y)} \ge e^{pf(x)} + e^{pf(z)}.$$

Subtracting 1 from both sides of the inequalities we obtain

$$e^{pf(x)} + e^{pf(y)} - 1 \ge e^{pf(x)} + e^{pf(z)} - 1$$

and taking logarithms from them we have

$$\ln[e^{pf(x)} + e^{pf(y)} - 1] \ge \ln[e^{pf(x)} + e^{pf(z)} - 1]$$

and

$$\frac{1}{p}\ln[e^{pf(x)} + e^{pf(y)} - 1] \ge \frac{1}{p}\ln[e^{pf(x)} + e^{pf(z)} - 1].$$

Taking into account that f^{-1} is a monotonically decreasing function, the following inequality is true

$$f^{-1}\{\frac{1}{p}\ln[e^{pf(x)} + e^{pf(y)} - 1]\} \le f^{-1}\{\frac{1}{p}\ln[e^{pf(x)} + e^{pf(z)} - 1]\}.$$

Therefore

$$h(x,y) \le h(x,z).$$

4. Boundary conditions.

Expression (28) at $y = 1$ takes the form

$$h(x,\,1) = f^{-1}\{\frac{1}{p}\ln[e^{pf(x)} + e^{pf(1)} - 1]\}.$$

Since $f(1) = 0$, we obtain for the method (28)

$$h(x,\,1) = f^{-1}\{\frac{1}{p}\ln[e^{pf(x)} + 1 - 1]\} = x.$$

Similarly, for $y = 0$ from the expression (28) we obtain

$$h(x,0) = \lim_{y\to 0} f^{-1}\{\frac{1}{p}\ln[e^{pf(x)} + e^{pf(0)} - 1]\} = \lim_{z\to\infty} f^{-1}(z) = 0.$$

Let's show that the generating method (28) forms Archimedean triangular norm. Let $y = x$. Then from the expression (28) we obtain

$$h(x,x) = f^{-1}\{\frac{1}{p}\ln[e^{pf(x)} + e^{pf(x)} - 1]\} = f^{-1}\{\frac{1}{p}\ln[2e^{pf(x)} - 1]\}.$$

At $x > 0$ and $p > 0$ we obtain $2e^{pf(x)} - 1 > e^{pf(x)}$, therefore

$$f^{-1}\{\frac{1}{p}\ln[2e^{pf(x)} - 1]\} < x.$$

Simultaneously $h(x, y)$ from (28) is monotone function, therefore this triangular norm is Archimedean.

Consequently, the generated by proposed method operator (28) satisfies the requirements of these axioms. Note, that this expression generates Archimedean triangular norms. Let us present some examples of the obtained by proposed method triangular norms.

6. Examples of operators received by a new generating method

Taking the generating method (28) as a means of construction of logical connective operators, we will build some of them.

Example 1. Generator $f(x) = -\ln(x)$. Then the inverse function is $f^{-1}(x) = \exp(-x)$. Substituting them in the expression (28) we obtain the well-known H. Hamacher triangular norm (17) and conorm (18) respectively

$T_1(x,y) = xy/(x+y-xy) = T_H(x,y), S_1(x,y) = (x+y-2xy)/(1-xy) = S_H(x,y).$

Example 2. Generator $f(x) = (1-x)/x$. Then the inverse function is $f^{-1}(x) = 1/(1+x)$. Substituting them in the expression (28) we obtain a new triangular norm

$$T_2(x,y) = \frac{p}{p + \ln[\exp(p\frac{1-x}{x}) + \exp(p\frac{1-y}{y}) - 1)]}, \tag{29}$$

and conorm

$$S_2(x,y) = 1 - \frac{p}{p + \ln[\exp(\frac{px}{1-x}) + \exp(p\frac{py}{1-y}) - 1)]}. \tag{30}$$

Fig. 1 shows the hyper surfaces of functions (29) (a) and (30) (b) respectively.

Example 3. Generator $f(x) = -1/\ln(1-x)$. Then the inverse function is $f^{-1}(x) = 1 - \exp(-1/x)$. Substituting them in the expression (28) we obtain a new triangular norm

$$T_3(x,y) = 1 - \exp(-\frac{p}{\ln(\exp(-p/\ln(1-x)) + \exp(-p/\ln(1-y)) - 1)}) \tag{31}$$

and conorm

$$S_3(x,y) = \exp(-\frac{p}{\ln(\exp(-p/\ln(x)) + \exp(-p/\ln(y)) - 1)}). \tag{32}$$

Fig. 2 shows the hyper surfaces of functions (31) (a) and (32) (b).

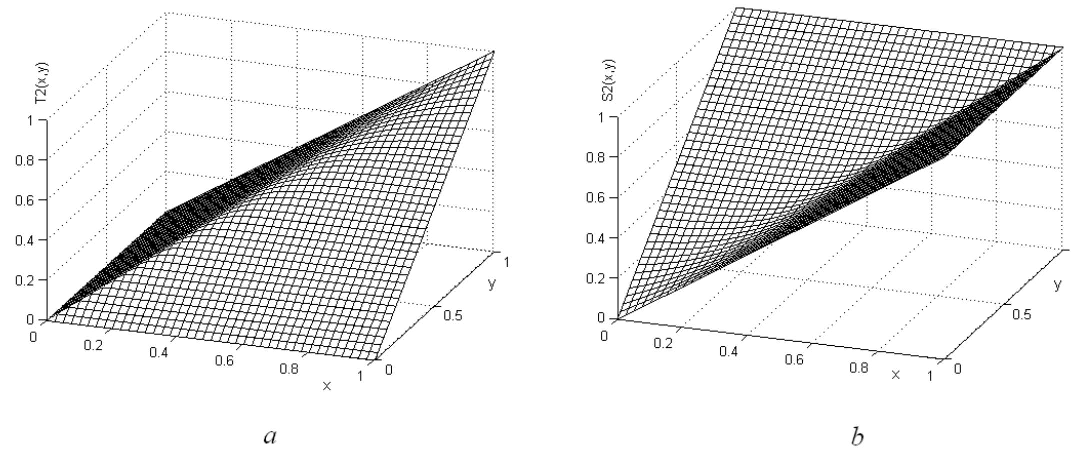

Fig. 1. The hyper surfaces of functions (29) (a) and (30) (b) at $p = 0,5$

Example 4. Generator $f(x) = (1 - \ln(x))^\alpha - 1$. Then the inverse function is $f^{-1}(x) = \exp[1 - (1+x)^{1/\alpha}]$. Substituting them in the expression (28) we obtain a new triangular norm

$$T_4(x,y) = \exp(1 - (1 + \frac{1}{p}\ln(\exp(p((1-\ln(x))^\alpha - 1) + \exp(p((1-\ln(y))^\alpha - 1) - 1))^{1/\alpha})$$

and conorm

$$S_4(x,y) =$$

$$= 1 - \exp(1 - (1 + \frac{1}{p}\ln(\exp(p((1-\ln(1-x))^\alpha - 1) + \exp(p((1-\ln(1-y))^\alpha - 1) - 1))^{1/\alpha}).$$

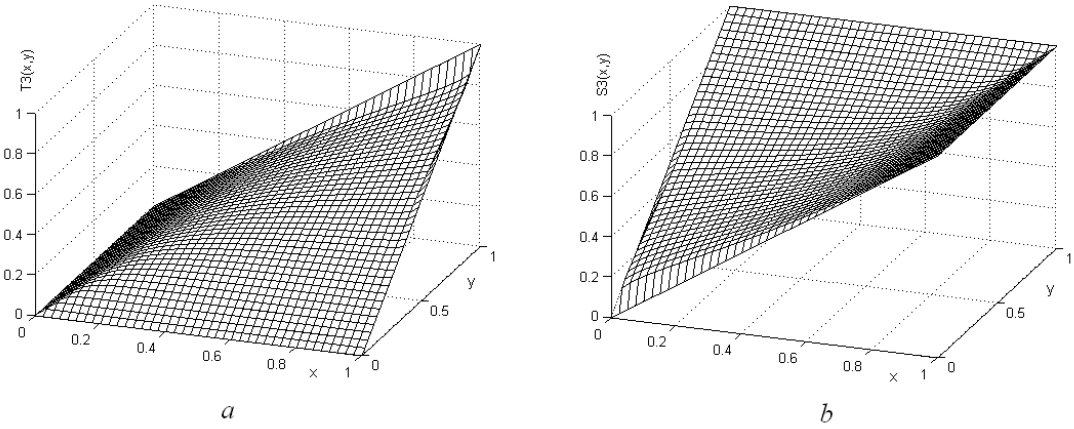

Fig. 2. Hyper surfaces of functions (31) a and (32) b at $p = 0,5$

Example 5. Generator $f(x) = (-\ln(x))^{\alpha}$. Then the inverse function is $f^{-1}(x) = \exp(-x^{1/\alpha})$. Substituting them in the expression (28) we obtain a new triangular norm

$$T_5(x, y) = \exp(1 - (\frac{1}{p} \ln(\exp(p(-\ln(x))^{\alpha}) + \exp(p(-\ln(y))^{\alpha}) - 1))^{1/\alpha}) \qquad (33)$$

and conorm

$$S_5(x, y) = \qquad (34)$$

$$= 1 - \exp(1 - (\tfrac{1}{p} \ln(\exp(p(-\ln(1-x))^{\alpha}) + \exp(p(-\ln(1-y))^{\alpha}) - 1))^{1/\alpha}).$$

Fig. 3 shows the hyper surfaces of functions (29) (a) and (30) (b).

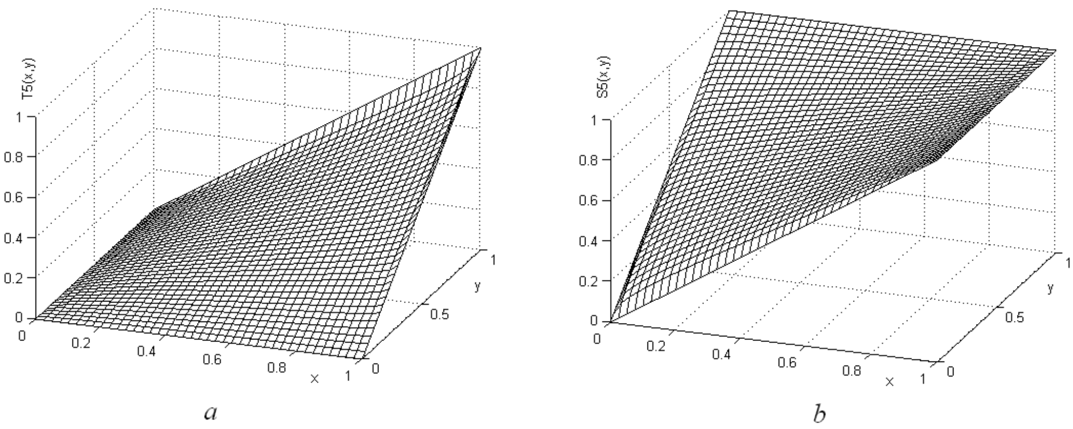

Fig. 3. Hyper surfaces of functions (33) $a$and (34) b at $p = 2$ and $\alpha = 0,4$

The given above figures demonstrate the character of the change of the logical connective operators hyper surface, which are represented by the corresponding t-

norms and t-conorms. Thus fig. 1-a illustrates the property of the hyper surface characteristic, that, with increasing of p, the function (29) approaches the $\min(x, y)$ as a limiting function gradually from small values of components x and y to large ones. The same property is illustrated by fig. 2-a but at decreasing of parameter p value. For the triangular norms obtained with the use of the classical generator (24), the character of such an approximation is close to the uniform across the range of values of the components x and y.

Conclusions

A new method for generating of logical connective operators for fuzzy systems is presented. It is shown that using it, is possible to construct new logical connectives with characteristics, which are conditioned by the choice of the value of the control parameter. Therefore, the fact that connectives are described by a parameterized function makes them more flexible. This will increase the effectiveness and extend the functionality of fuzzy logic systems.

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METODA GENEROWANIA ARCHIMEDESOWYCH TRÓJKĄTNYCH OPERATORÓW POŁĄCZENIOWYCH

S t r e s z c z e n i e

Trójkątne operatory Archimedesowe stanowią podstawę do opracowywania, analizy i projektowania systemów rozmytych. Są generowane przez proste, monotoniczne, jednoznaczne i ciągłe funkcje. Analizowane są podstawowe zestawy połączeń logicznych systemów rozmytych. Pokazano, że są one podzielone na warunkowe i algebraiczne. Opisano dobrze znane metody generowania trójkątnych operatorów Archimedesowych. Wyodrębniono ograniczenia charakterystyk funkcjonalnych tak generowanych operatorów. Aby rozszerzyć charakterystyki operatorów, stworzono metodę generowania, która umożliwia budowanie nowych sparametryzowanych operatorów. Podano przykłady konstruowania zarówno nowych operatorów, jak i już znanych. Opisana tendencja zmiany charakterystycznej hiperppowierzchni, którą buduje nowy operator.

Słowa kluczowe: T-normy, S-normy, T-operatory, generatory połączeń