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$Sp(n)$-ORBITS IN THE GRASSMANNIANS OF COMPLEX AND Σ-COMPLEX SUBSPACES OF AN HERMITIAN QUATERNIONIC VECTOR SPACE

Abstract

We determine the invariants characterizing the $Sp(n)$-orbits in the real Grassmannian $Gr^{\mathbb{R}}(2k, 4n)$ of the $2k$-dimensional complex and Σ-complex subspaces of a $4n$-dimensional Hermitian quaternionic vector space. A Σ-complex subspace is the orthogonal sum of complex subspaces by different, up to sign, compatible complex structure. The result is obtained by considering two main features of such subspaces. The first is that any such subspace admits a decomposition into an Hermitian orthogonal sum of 4-dimensional complex addends plus a 2-dimensional totally complex subspace if k is odd, meaning that the quaternionification of the addends are orthogonal in pairs. The second is that any 4-dimensional complex addend U is an isoclinic subspace i.e. the principal angles of the pair (U, AU) are all the same for any compatible complex structure A. Using these properties we determine the full set of the invariants characterizing the $Sp(n)$-orbit of such subspaces in $Gr^{\mathbb{R}}(2k, 4n)$.

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Summary

In this paper we study the $Sp(n)$-orbits in the real Grassmannians $Gr^{\mathbb{R}}(2k, 4n)$ of some special $2k$-dimensional subspaces of a $4n$-dimensional real vector space V. We endow V with an Hermitian quaternionic structure $(\mathcal{Q}, <, >)$, an Hermitian product $(\cdot)$ and denote by $S(\mathcal{Q})$ the 2-sphere of complex structures $J \in \mathcal{Q}$. The invariants characterizing the $Sp(n)$-orbit of a generic subspace $U \subset V$ appear in [20] where we find some equivalent statements. The first one we report here is given in the Theorem (3.1) where it is stated that a pair of subspaces U and W of real dimension m in the $\mathbb{H}$-module V^{4n} belong to the same $Sp(n)$-orbit iff there exist bases $\mathcal{B}_U = (X_1, \dots, X_m)$ and $\mathcal{B}_W = (Y_1, \dots, Y_m)$ of U and W respectively w.r.t. which for the Hermitian products one has $(X_i \cdot X_j) = (Y_i \cdot Y_j),\ i = 1, \dots, m$ for one and hence any (hypercomplex) admissible basis of $\mathcal{Q}$.

Let $\boldsymbol{\theta}^A(U)$ be the vector of the principal angles between the pair $(U, AU),\ A \in S(\mathcal{Q})$ in non decreasing order. A consequence of the Theorem (3.1) is that a necessary condition for U and W to share the same $Sp(n)$-orbit is that, for one and hence any admissible basis (I, J, K), one has $\boldsymbol{\theta}^I(U) = \boldsymbol{\theta}^I(W),\ \boldsymbol{\theta}^J(U) = \boldsymbol{\theta}^J(W),\ ,\boldsymbol{\theta}^K(U) = \boldsymbol{\theta}^K(W)$.

The determination of the principal angles between a pair of subspaces S, T is a well know problem solved by the singular value decomposition of the orthogonal projector of S onto T. Here, for a chosen $U \subset V$ we consider the pairs $(U, AU), A \in S(\mathcal{Q})$ and denote by Pr^{AU} the orthogonal projector of U onto AU. In this case the singular values of Pr^{AU} are always degenerate which implies that they have non-unique singular vectors. In terms of principal vectors of the pair (U, AU) we can equivalently say that the principal vectors are never uniquely defined.

Another way to obtain the principal angles and the associated principal vectors between the pair of subspaces (U, AU), for any $A \in S(\mathcal{Q})$, is through the standard decompositions of the restriction to U of the A-Kähler skew-symmetric form $\omega^A : (X, Y) \mapsto < X, AY >,\ X, Y \in U$. Calling *standard basis* any orthonormal basis w.r.t. which $\omega^A|_U$ assumes standard form, with the non-negative entries ordered in non increasing order, and denoting by $\mathcal{B}^A(U)$ the set of all such bases one has that any $B \in \mathcal{B}^A(U)$ consists of principal vectors of the pair (U, AU). As stated beforehand, for any $A \in S(\mathcal{Q})$ the standard bases of ω^A are never unique not even if U is 2-dimensional. This is the main problem in determining the $Sp(n)$-orbits in $Gr^{\mathbb{R}}(k, 4n)$ as it is evident from an equivalent conditions to the one stated in Theorem (3.1) appearing in [20] and here reported in Theorem (3.2). According to it, a pair of subspaces U, W are in the same $Sp(n)$-orbit iff, for one and hence any admissible basis (I, J, K) of $\mathcal{Q}$ the following 2 conditions are satisfied:

- $\boldsymbol{\theta}^I(U) = \boldsymbol{\theta}^I(W),\ \boldsymbol{\theta}^J(U) = \boldsymbol{\theta}^J(W),\ \boldsymbol{\theta}^K(U) = \boldsymbol{\theta}^K(W)$;
- there exist three orthonormal bases $(\{X_i\}, \{Y_i\}, \{Z_i\}) \in (\mathcal{B}^I(U) \times \mathcal{B}^J(U) \times \mathcal{B}^K(U))$ and $(\{X'_i\}, \{Y'_i\}, \{Z'_i\}) \in (\mathcal{B}^I(W) \times \mathcal{B}^J(W) \times \mathcal{B}^K(W))$ whose relative position is the same or equivalently

$$A = A', \qquad B = B'$$

  where $A = (< X_i, Y_j >), \qquad A' = (< X'_i, Y'_j >), \qquad B = (< X_i, Z_j >), B' = (< X'_i, Z'_j >)$.

The problem to determine the $Sp(n)$-orbits turns therefore into the one of determining the existence of such triple of bases. In [20] we set up a procedure to determine a triple of *canonical bases* w.r.t. which one computes the matrices A and B. Namely, fixed an admissible basis (I, J, K), a triple of canonical bases of a subspace U is constituted by a triple of standard bases of $\omega^I, \omega^J, \omega^K$ that either are uniquely determined by the procedure aforementioned or, if this is not the case, nevertheless the associated matrices A, B are unique.

We call the matrices obtained thereof *canonical matrices* and denote them by C_{IJ} and C_{IK}. In [20], Chosen an admissible basis (I, J, K) we associate then to any subspace $U \subset V$ the following invariant

$$Inv(U) = \{\boldsymbol{\theta}^I(U), \boldsymbol{\theta}^J(U), \boldsymbol{\theta}^K(U), C_{IJ}, C_{IK}\},$$

and in the Theorem (3.3) we affirm that the subspaces U and W of V are in the same $Sp(n)$-orbit iff $Inv(U) = Inv(W)$. If $Inv(U) = Inv(W)$ w.r.t. the admissible basis (I, J, K) then $Inv(U) = Inv(W)$ for any admissible basis.

After recalling the definition of some special subspaces of V, in Proposition (4), we show that a generic subspace U of $(V, \mathcal{Q}, <,>)$ admits a decomposition

$$U^m = U_Q \overset{\perp}{\oplus} U^\Sigma \overset{\perp}{\oplus} U_R \qquad \text{with} \qquad U^\Sigma = (U_1, I_1) \overset{\perp}{\oplus} \dots \overset{\perp}{\oplus} (U_p, I_p)$$

into an orthogonal sum of the maximal quaternionic subspace U_Q, a Σ-complex subspace U^Σ, defined as the orthogonal sum of maximal I_i-complex subspaces (U_i, I_i) with $I_i \in S(\mathcal{Q})$ and a totally real subspace U_R. In Proposition (1.11) we prove that the complex addends U_Q and (U_i, I_i) are Hermitian orthogonal i.e. their quaternionifications U_Q and $(U_i)^{\mathbb{H}}$ are orthogonal in pairs. In general this is not true for the orthogonal totally real addend U_R.

The subspaces we consider in this paper are the I-complex subspaces (U, I) with $I \in S(\mathcal{Q})$ and the Σ-complex subspaces. It will turn out that the analysis of the 4-dimensional complex case is fundamental to determine the $Sp(n)$-orbit of such subspaces. Any 4-dimensional I-complex subspace (U, I) is characterized by the fact that, for any $A \in S(\mathcal{Q})$ the pair (U, AU) is isoclinic. In [21] we denoted by $\mathcal{IC}^4$ the set of all 4-dimensional subspaces sharing this property and we called them *isoclinic subspaces*. We then recall the main results which concern a subspace $U \in \mathcal{IC}^4$ referring to [21] for proofs and a wider treatment.

In [21] we determined the triple of canonical bases $\{X_i\}, \{Y_i\}, \{Z_i\}$, $i = 1, \dots, 4$ of U. Given an admissible basis (I, J, K) and chosen $X_1 \in U$, we considered the standard 2-planes $U^I = L(X_1, X_2)$, $U^J = L(X_1, Y_2)$, $U^K = L(X_1, Z_2)$ centered on a common unitary vector X_1 of the skew-symmetric forms $\omega^I, \omega^J, \omega^K$ respectively where

$$X_2 = \frac{I^{-1} Pr_U^{IU} X_1}{\cos \theta^I}, \quad Y_2 = \frac{J^{-1} Pr_U^{JU} X_1}{\cos \theta^J}, \quad Z_2 = \frac{K^{-1} Pr_U^{KU} X_1}{\cos \theta^K}.$$

Denoted by $\xi =< X_2, Y_2 >$, $\chi =< X_2, Z_2 >$, $\eta =< Y_2, Z_2 >$, in Corollary (3.11) we proved that the triple (ξ, χ, η) is an invariant of U. We introduced Γ as a function of (ξ, χ, η) and $\Delta = \pm\sqrt{1 - \Gamma^2}$. After proving that the pair (Γ, Δ) itself is an invariant of U, in the Proposition (3.21) we affirm that the invariants $(\xi, \chi, \eta, \Delta)$ determine the canonical matrices C_{IJ} and C_{IK} which are given in (21) w.r.t. the canonical bases. Therefore, according to the statement of the Theorem (3.3), in the Theorem (3.22) we state that the invariants $(\xi, \chi, \eta, \Delta)$ together with the angles $(\theta^I, \theta^J, \theta^K)$ determine the $Sp(n)$-orbit of any $U \in \mathcal{IC}^4$ in $G^{\mathbb{R}}(4, 4n)$.

The set of 4-dimensional complex subspaces is a subset of $\mathcal{IC}^4$. Let then (U, I) be a 4-dimensional I-complex subspace with $I \in S(\mathcal{Q})$. Fixed an adapted basis

(I,J,K), we associate to (U,I) the $I^{\perp}$- Kähler angle $\theta^{I^{\perp}}$ which is one of the four identical principal angles of the pair (U,KU) observing that KU is the same for any $K\in I^{\perp}\cap S(\mathcal{Q})$. In this case the angles of isoclinicity $(\theta^I,\theta^J,\theta^K)=(0,\theta^{I^{\perp}},\theta^{I^{\perp}})$ and we denote such subspace by the triple $(U,I,\theta^{I^{\perp}})$. Furthermore, considered a triple of canonical bases $\{X_i\},\{Y_i\},\{Z_i\}, i=1,\dots,4$ of the skew-symmetric forms $\omega^I,\omega^J,\omega^K$ centered on X_1 one has $\xi=\chi=\eta=0$ and the matrices C_{IJ} and C_{IK} of all 4-dimensional complex subspaces are given in (28). Then, according to the Theorem (3.3), the pair $(I,\theta^{I^{\perp}})$ determines the $Sp(n)$-orbit of U i.e. all and only the 4-dimensional I-complex subspaces with same $I^{\perp}$- Kähler angle constitute one $Sp(n)$-orbit in $G^{\mathbb{R}}(4,4n)$ as stated in Theorem (3.33).

In Theorem (3.37) we affirm that a $2m$-dimensional I-complex subspace admits an Hermitian orthogonal decomposition into 4-dimensional I-complex subspaces. Although such decomposition is not unique we can associate to U the canonically defined $I^{\perp}$**-Kähler multipleangle** $\boldsymbol{\theta^{I^{\perp}}}=(\theta_1^{I^{\perp}},\dots,\theta_{[m/2]}^{I^{\perp}})$ ($\boldsymbol{\theta^{I^{\perp}}}=(\theta_1^{I^{\perp}},\dots,\theta_{m/2}^{I^{\perp}},\pi/2)$ if m is odd) of the I-complex $2m$-dimensional subspace (U,I) being $\boldsymbol{\theta}^{I^{\perp}}$ the set of the $I^{\perp}$-Kähler angle of the Hermitian orthogonal 4-dimensional I-complex addends (plus the K-Kähler angle of an Hermitian orthogonal totally I-complex plane if m is odd with $K\in I^{\perp}$). Denoted by $Gr^{\mathbb{R}}_{(I,\boldsymbol{\theta}^{I^{\perp}})}(2m,4n)$ the set of $2m$-dimensional pure I-complex subspaces in $(V^{4n},<,>,\mathcal{Q})$ with $I^{\perp}$-Kähler multipleangle $\boldsymbol{\theta^{I^{\perp}}}$, in Theorem (3.42) we state that the group $Sp(n)$ acts transitively on $Gr^{\mathbb{R}}_{(I,\boldsymbol{\theta}^{I^{\perp}})}(2m,4n)$ i.e. the pair $(I,\boldsymbol{\theta}^{I^{\perp}})$ composed by the complex structure $I\in\mathcal{Q}$ and the $I^{\perp}$-Kähler multipleangle $\boldsymbol{\theta}^{I^{\perp}}$ of the I-complex subspace U determines completely its $Sp(n)$-orbit in the Grassmannian $Gr^{\mathbb{R}}(2m,4n)$.

In particular, all totally I-complex subspaces of same dimension form one orbit in $G^{\mathbb{R}}(2m,4n)$.

We then consider a Σ-complex subspace U. From Proposition (1.14) the decomposition of U into an orthogonal sum of maximal pure complex subspaces by different (up to sign) structures is unique. Moreover, from Proposition (1.11), the $2m_i$-dimensional I_i-complex subspaces are Hermitian orthogonal and to determine the $Sp(n)$-orbit we can deal separately with each I_i-complex addend. In Proposition (3.43) we state that the pair $(\mathcal{I},\Theta)$, where $\mathcal{I}=\{I_i\}$ and $\Theta=\{\boldsymbol{\theta_i^{I_i^{\perp}}}\}$ is the vector whose elements are the $I_i^{\perp}$-Kähler multipleangle of the I_i-complex addend, completely determines the orbit in the real Grassmannian.

1 Decomposition of a generic subspace of an Hermitian quaternionic vector space

1.1 The Hermitian quaternionic structure

Let V be a real vector space of dimension $4n$.

Definition 1.1.

1. *A triple $\mathcal{H} = \{J_1, J_2, J_3\}$ of anticommuting complex structures on V with $J_1 J_2 = J_3$ is called a* ***hypercomplex structure*** *on V.*

2. *The 3-dimensional subalgebra*

$$\mathcal{Q} = span_{\mathbb{R}}(\mathcal{H}) = \mathbb{R}J_1 + \mathbb{R}J_2 + \mathbb{R}J_3 \approx \mathfrak{sp}_1$$

   of the Lie algebra $End(V)$ is called a ***quaternionic structure*** *on V.*

Note that two hypercomplex structures $\mathcal{H} = \{J_1, J_2, J_3\}$ and $\mathcal{H}' = \{J_1', J_2', J_3'\}$ generate the same quaternionic structure $\mathcal{Q}$ iff they are related by a rotation, i.e.

$$J'_\alpha = \sum_\beta A^\beta_\alpha J_\beta, \qquad (\alpha = 1, 2, 3)$$

with $(A^\beta_\alpha) \in SO(3)$. A hypercomplex structure generating $\mathcal{Q}$ is called an **admissible basis** of $\mathcal{Q}$. We denote by $S(\mathcal{Q})$ the 2-sphere of complex structures $J \in \mathcal{Q}$ i.e. $S(\mathcal{Q}) = \{aJ_1 + bJ_2 + cJ_3,\ a, b, c \in \mathbb{R},\ a^2 + b^2 + c^2 = 1\}$.

A real vector space V endowed with a hypercomplex structure (J_1, J_2, J_3) is an $\mathbb{H}$-module by defining scalar multiplication by a quaternion q as follows:

$$qX = (a + ib + jc + dk)X = aX + bJ_1X + cJ_2X + dJ_3X, \quad X \in V,\ a, b, c, d, \in \mathbb{R}$$

and (i, j, k) a basis of $Im(\mathbb{H})$ satisfying

$$i^2 = j^2 = k^2 = -1;\ ij = -ji = k. \tag{1}$$

Definition 1.2. *An Euclidean scalar product $<,>$ in V is called* ***Hermitian*** *w.r.t. a hypercomplex basis $\mathcal{H} = (J_\alpha)$ (resp. the quaternionic structure $\mathcal{Q} = span(\mathcal{H})_{\mathbb{R}}$) iff for any $X, Y \in V$*

$$< J_\alpha X, J_\alpha Y > = < X, Y > \text{ or equivalently } < J_\alpha X, Y > = - < X, J_\alpha Y >, \qquad (\alpha = 1, 2, 3)$$

(respectively,

$$< JX, JY > = < X, Y > \text{ or equivalently } < JX, Y > = - < X, JY >, \quad (\forall J \in \mathcal{Q}))$$

Definition 1.3. *A hypercomplex structure $\mathcal{H}$ (resp. quaternionic structure $\mathcal{Q}$) together with an Hermitian scalar product $<,>$ is called an* ***Hermitian hypercomplex*** *(resp.* ***Hermitian quaternionic) structure*** *on V and the triple $(V^{4n}, \mathcal{H}, <,>)$ (resp. $(V^{4n}, \mathcal{Q}, <,>)$) is an* ***Hermitian hypercomplex*** *(resp.* ***quaternionic) vector space****.*

The prototype of an Hermitian hypercomplex vector space is the n-dimensional quaternionic numerical space $\mathbb{H}^n$ which is a real vector space of dimension $4n$, a $\mathbb{H}$-module with respect to right (resp. left) multiplication by quaternions and is endowed with the canonical positive definite Hermitian product

$$\mathbf{h} \cdot \mathbf{h}' = \sum_{\alpha=1}^n \overline{h_\alpha} h'_\alpha \quad (\text{resp. } \mathbf{h} \cdot \mathbf{h}' = \sum_{\alpha=1}^n h_\alpha \overline{h'_\alpha}) \qquad \mathbf{h} = (h_1, \ldots, h_n),\ \mathbf{h}' = (h'_1, \ldots, h'_n) \in \mathbb{H}^n. \tag{2}$$

The real part of the Hermitian product defines an Euclidean scalar product $<,>= Re(\cdot)$ on the real vector space $\mathbb{H}^n \simeq \mathbb{R}^{4n}$. If we consider the basis $(1, i, j, k)$ of $\mathbb{H}$ satisfying the multiplication table obtainable from the conditions (1) one has that the right multiplications by $-i, -j, -k$ (resp. left multiplication by i, j, k) induce real endomorphisms $(I = R_{-i}, J = R_{-j}, K = R_{-k})$ (resp. $(I = L_i, J = L_j, K = L_k)$) of the $\mathbb{H}$-module $\mathbb{H}^n$ satisfying $I^2 = J^2 = K^2 = -Id$, $IJ = K = -JI$ and skew-symmetric with respect to the metric $<,>$ i.e. an Hermitian hypercomplex structure on $\mathbb{H}^n$.

We recall that a new basis $(1, i', j', k')$ of $\mathbb{H}$ give rise to the multiplication table (1) iff $(i', j', k') = (i, j, k)C$ with $C \in SO(3)$. We will denote by $\mathcal{B}$ the set of bases of $\mathbb{H}$ satisfying the relations (1) and call it **canonical system of bases**. In [3] it has been proved that

Proposition 1.4. *[3] Both the Hermitian product and the scalar product of $\mathbb{H}^n$ have intrinsic meaning (SPIEGARE) with respect to the canonical system of bases $\mathcal{B}$.*

Let $(1, i, j, k) \in \mathcal{B}$ be a chosen basis in $\mathbb{H}$ and denote by $I = R_{-i},\ J = R_{-j},\ K = R_{-k}$ the real endomorphisms of the $\mathbb{H}$-module $\mathbb{H}^n$. Let $\mathcal{Q} = span_{\mathbb{R}}(I, J, K)$.

Proposition 1.5. *For the scalar product and the Hermitian product of a pair of vectors $L, M \in \mathbb{H}^n$ the following relation holds:*

$$L \cdot M =< L, M > + < L, IM > i+ < L, JM > j+ < L, KM > k \tag{3}$$

Proof. We prove that $< L, IM >, < L, JM >, < L, KM >$ are respectively the coefficients of i, j, k in the Hermitian product $L \cdot M$. In fact $< L, IM >= Re(L \cdot -Mi) = -Re(L \cdot M)i$ which is exactly the coefficient of i of the quaternion $L \cdot M$ and analogously for $< L, JM >$ and $< L, KM >$. □

After identifying an admissible hypercomplex structures (I, J, K) of V with (R_{-i}, R_{-j}, R_{-k}) of $\mathbb{H}$, we can endow a quaternionic Hermitian vector space with the Hermitian product given in (3). It has an intrinsic meaning w.r.t. the admissible bases, that is, using a different admissible basis $(I', J', K') = (I, J, K)C,\ C \in SO(3)$, the coordinates of the obtained quaternion are w.r.t. the basis $(i', j', k') = (i, j, k)C$. In the following we consider such an identification.

1.2 Special subspaces of an Hermitian quaternionic vector space

Let $(V^{4n}, \mathcal{Q}, <,>)$ be an Hermitian quaternionic vector space endowed with the Hermitian product given in (3). In the following, given a finite set $\{M_1, \dots, M_s\}$ of vectors of V, we denote by $L(M_1, \dots, M_s)$ or equivalently by $span_{\mathbb{R}}(M_1, \dots, M_s)$ their linear span over $\mathbb{R}$. We will denote by (I, J, K) a generic admissible basis of $\mathcal{Q}$ and by $I^{\perp} = L(J, K) \cap S(\mathcal{Q})$ i.e. $I^{\perp} = \{\beta J + \gamma K,\ \beta^2 + \gamma^2 = 1\}$. Moreover, given a subspace $U \subset V$, $U^{\mathbb{H}}$ denotes its quaternionification i.e. the subspace spanned on $\mathbb{H}$ by some basis of U. In particular, given a vector $X \in V$ by $\mathcal{Q}X$ we denote the 4-dimensional real subspace real image of the 1-dimensional subspace generated by X over $\mathbb{H}$ i.e. $\mathcal{Q}X = span_{\mathbb{R}}(X, IX, JX, KX) = (\mathbb{R}X)^{\mathbb{H}}$.

Definition 1.6. *Let $(V^{4n}, \mathcal{Q}, <,>)$ be an Hermitian quaternionic vector space. A subspace $U \subset V$ is* ***quaternionic*** *if $AU = U$, $\forall A \in S(\mathcal{Q})$. A subspace is* ***pure*** *if it does not contain any non trivial quaternionic subspace. Let $I \in S(\mathcal{Q})$, then U is* ***I-complex*** *and we denote it by (U, I) if $U = IU$. In particular (U, I) is a* ***totally I-complex*** *subspace if, for any $J \in I^{\perp}$, the pair of subspaces (U, JU) are strictly orthogonal* [1]*. A subspace is* ***totally real*** *if it does not contain any complex subspace or equivalently if $AU \cap U = \{0\}$, $\forall A \in S(\mathcal{Q})$. In particular it is a* ***r.h.p.s.*** *(real hermitian product subspace) if, for any pair of vectors $X, Y \in U$, one has $X \cdot Y \in \mathbb{R}$.*

A totally I-complex subspace is a **c.h.p.s.** (complex hermitian product subspace) i.e., for any pair of vectors $X, Y \in U$, $X \cdot Y = a + ib$, $a, b \in \mathbb{R}$. Clearly a 2-dimensional I-complex subspace is totally I-complex. Furthermore a totally real subspace is a r.h.p.s. iff for any admissible basis, the pairs (U, IU), (U, JU), (U, KU) are strictly orthogonal. We recall some results regarding the subspaces just defined:

Claim 1.7.

- *For quaternionic subspaces:*

1. *The sum and the intersection of quaternionic subspaces is a quaternionic subspace.*

   Proof. In fact if $0 \neq X \in U_1 \cap U_2$ with U_1, U_2 quaternionic subspaces then $\mathcal{Q}X \in U_1$ and $\mathcal{Q}X \in U_2$ then $\mathcal{Q}X \in U_1 \cap U_2$. For the sum, one has that for every vector $Z = X + Y \in U_1 + U_2$ and every $A \in S(\mathcal{Q})$ one has $AZ = AX + AY \in U_1 + U_2$. □

2. *The orthogonal complement of a quaternionic subspace of a quaternionic space is quaternionic.*

   Proof. Let U be quaternionic and $W \subset U$ quaternionic as well. Let consider the orthogonal decomposition $U = W \oplus W^{\perp}$. For $X \in W^{\perp}$, $A \in S(\mathcal{Q})$, let $AX = X_1 + X_2$ with $X_1 \in W = AW$ and $X_2 \in W^{\perp}$. Being $A(AX) = -X = AX_1 + AX_2$ it follows that $X_1 = \mathbf{0}$ and any $A \in S(\mathcal{Q})$ preserves $W^{\perp}$ i.e. $W^{\perp}$ is quaternionic. □

   - *For I-complex subspaces:*

3. *The subspace U is I-complex iff it is $(-I)$-complex. In the following, when we will speak of an I-complex subspace we will always imply "up to sign".*

[1] A pair of subspaces A, B are *orthogonal* if the angle between them is $\pi/2$ i.e. if there exists a line in A orthogonal to B. In particular they are *strictly orthogonal* and we wrote $A \perp B$ if any line in A is orthogonal to B. In terms of principal angles (whose definition we recall in (2.3)), we can say that a pair of subspaces A, B of dimensions m, n, $m \leq n$ is orthogonal if at least one of the principal angle is $\pi/2$ and is strictly orthogonal of all m principal angles equal $\pi/2$. Clearly the pair (A, B) is strictly orthogonal iff if is orthogonal and isoclinic (see the Definition (2.7). For instance, for any $T \in S(\mathcal{Q})$ and $U \subset V$ a 2-plane, the pair U, TU is isoclinic. Then in this case one can speak indifferently of orthogonality or strictly orthogonality.

4. *Let (U, I) be pure and suppose $(U', I') \subseteq U$. Then $I' = \pm I$. In other words an I-complex subspace does not contain any complex subspace by a different complex structure (up to sign).*

   Proof. Suppose $0 \neq Y \in U \cap U'$ then $I(I'Y) = I(\alpha IY + \beta JY + \gamma KY) = -\alpha Y + \beta KY - \gamma JY \in U$ which implies $\beta KY - \gamma JY = K(\beta Y - \gamma IY) \in U$. From pureness of U it follows $\beta = \gamma = 0$ i.e. $I' = \pm I$. $\square$

5. *The orthogonal complement to a complex subspace W of an I-complex space U is an I-complex subspace.*

   Proof. From previous statement, W is necessarily $\pm I$-complex. Consider then $(W, I) \subset (U, I)$ and the orthogonal decomposition $(U, I) = (W, I) \overset{\perp}{\oplus} \tilde{U}$ where $\tilde{U}$ is the orthogonal complement to W in U. Let $Z \in \tilde{U}$ non null and consider the vector $IZ = X + Y$ with $X \in W$ and $Y \in \tilde{U}$. Then $\tilde{U} \ni I(IZ) = IX + IY$ implies that $X = 0$ i.e. $\tilde{U}$ is I-complex. $\square$

6. *Sum and intersection of I-complex subspaces is an I-complex subspace.p*

   Proof. Let $W = (U_1, I) \cap (U_2, I)$. If $Z \in W$ then $IZ \in U_1$ and $IZ \in U_2$ then W is I-complex. Let now consider $U_1 + U_2 = \bar{U_1} \overset{\perp}{\oplus} W \overset{\perp}{\oplus} \bar{U_2}$ where $\bar{U_1}$ (resp. $\bar{U_2}$) is the orthogonal complement to W in U_1 (in U_2). From (5), the subspaces $\bar{U_1}$ and $\bar{U_2}$ are I-complex. If $T \in (U_1 + U_2) = X + Y + Z$ with $X \in \bar{U_1}$, $Y \in W$, $Z \in \bar{U_2}$ one has $IT \in U_1 + U_2$. $\square$

7. *Let (U, I) be an I-complex subspace. An* ***adapted basis*** *of U is an admissible basis containing I. For any $K, K' \in I^{\perp}$ one has that $KU = K'U$ is I-complex.*

   Proof. Let (I, J, K) be an adapted bases. One has $JU = KIU = KU$. Let $K' = \alpha J + \beta K \in I^{\perp}$. For any $X \in U$ it is $K'X = \alpha JX + \beta KX \in JU$. The subspace KU is clearly I-complex since $IKU = KIU = KU$. $\square$

8. *If (U, I) is pure, then $JU \cap U = \{0\}$ for any $J \in I^{\perp}$.*

   Proof. In fact suppose $JY \in U \cap JU$ with $Y \in U$. Then Y, IY, JY are in U as well as $I(JY) = KY$ which is absurd by the hypothesis of pureness. $\square$

9. *The intersection of a pair of pure complex subspaces by different, up to sign, complex structures is a totally real subspace.*

   Proof. Let (U, I) and (U', I') be a pair of pure complex subspaces with $I' \neq \pm I$ and denote $W = U \cap U'$. Suppose W is complex. Applying the previous result, W as a subspace of U can only be a pure $\pm I$-complex and as a subspace of U' can only be pure $\pm I'$-complex. Then W is totally real. $\square$

- *For totally real subspaces:*

10. *By definition, a totally real subspace U is pure.*

11. *Given a hypercomplex basis (I, J, K) it is $IU \cap JU = IU \cap KU = JU \cap KU = \{0\}$.*

    Proof. In fact, suppose $0 \neq Y = Iv = Jw \in IU \cap JU$ for the non null vectors, $v, w \in U$. Then $I^2 v = -v = IJ(w) = Kw$ which is absurd since by definition $U \cap KU = \{0\}$. $\square$

12. *Let U be a totally real subspace. If U is a r.h.p. subspace* $\dim(U)^{\mathbb{H}} = 4 \dim U$ *otherwise* $\dim(U)^{\mathbb{H}} \geq 2 \dim U$.

    Proof. If U is a r.h.p.s. for any admissible basis (I, J, K), any pair of the 4 subspaces (U, IU, JU, KU) is strictly orthogonal. If instead U is totally real then clearly $\dim(U)^{\mathbb{H}} \geq 2 \dim(U)$ since, from (11), any pair of the subspaces U, IU, JU, KU have trivial intersection. $\square$

13. *Any subspace of a r.h.p. subspace is a r.h.p.s.*

    Proof. It's straightforward. $\square$

1.3 Decomposition of a generic subspace

The following proposition shows that, by using quaternionic, pure complex and totally real subspaces as building blocks, we can build up any subspace U of an Hermitian quaternionic vector space $(V^{4n}, \mathcal{Q}, <,>))$.

Proposition 1.8. *Let $U \subseteq V$ be a subspace and let U_Q be its maximal quaternionic subspace. Then U admits an orthogonal decomposition of the form*

$$\begin{aligned} U^m &= U_Q \overset{\perp}{\oplus} U^{\Sigma} \overset{\perp}{\oplus} U_R \qquad\qquad \textit{with} \\ U^{\Sigma} &= (U_1, I_1) \overset{\perp}{\oplus} \ldots \overset{\perp}{\oplus} (U_p, I_p) \end{aligned} \tag{4}$$

where (U_i, I_i) are maximal pure I_i-complex and U_R is totally real.

Proof. For any $A \in S(\mathcal{Q})$ we denote by U_A the maximal A-invariant subspace in U. Let U^1 be the orthogonal complement to U_Q in U and choose a complex structure I_1 such that $(U_1, I_1) := U^1_{I_1} \neq \{0\}$. Then we can write $U = U_Q \overset{\perp}{\oplus} (U_1, I_1) \overset{\perp}{\oplus} U^2$ where U^2 is the orthogonal complement in U to $U_Q \overset{\perp}{\oplus} (U_1, I_1)$.

Let now choose a complex structure I_2 such that $(U_2, I_2) := U^2_{I_2} \neq \{0\}$. Then $U = U_Q \overset{\perp}{\oplus} (U_1, I_1) \overset{\perp}{\oplus} (U_2, I_2) \overset{\perp}{\oplus} U^3$ where U^3 is the orthogonal complement to $U_Q \overset{\perp}{\oplus} (U_1, I_1) \overset{\perp}{\oplus} (U_2, I_2)$.

Denote by $p+1$ the step in which U^{p+1} has no invariant complex subspace. Then $U^{p+1} = U_R$ is totally real. $\square$

Proposition 1.9. *It is* $(U_R)^{\mathbb{H}} \cap U_Q = \{0\}$.

Proof. Suppose $W = (U_R)^{\mathbb{H}} \cap U_Q$. From point (1) of the Claim (1.7), W is quaternionic. Let $(U_R)^{\mathbb{H}} = W \oplus W^\perp$. From point (2) of the same Claim, $W^\perp$ is quaternionic and since $U_R \subset W^\perp$ being by construction $U_Q \perp U_R$ one has that $(U_R)^{\mathbb{H}} = W^\perp$ i.e. $W = \{\mathbf{0}\}$. □

In the following, given a pair of subspaces (A, B), by $A \overset{H}{\perp} B$ we intend that A and B are orthogonal in Hermitian sense i.e. their quaternionifications $A^{\mathbb{H}}, B^{\mathbb{H}}$ are strictly orthogonal (in other words $A \overset{H}{\perp} B$ is equivalent to $A^{\mathbb{H}} \perp B^{\mathbb{H}}$). We now prove the following facts:

1. In the orthogonal sum $U_Q \overset{\perp}{\oplus} (U_1, I_1) \overset{\perp}{\oplus} \dots \overset{\perp}{\oplus} (U_p, I_p)$ any pair of the complex addends are orthogonal in Hermitian sense.
2. If in (1.8) one has $U_R = \{0\}$ the given decomposition is unique.

To prove (1) we need the following

Lemma 1.10. *Let* $U_1 = L(X, IX)$ *and* $U_2 = L(Y, I'Y)$ *be a pair of 2-dimensional complex subspaces . Then if* $I' \neq \pm I$ *we have that*

$$U_1 \perp U_2 \Leftrightarrow U_1 \overset{H}{\perp} U_2, \quad (\textit{ i.e.} \Leftrightarrow \mathcal{Q}X \perp \mathcal{Q}Y).$$

Proof. Since for any complex structure $A \in S(\mathcal{Q})$, the 2-plane $L(X, AX) \subset \mathcal{Q}X$, then clearly if $\mathcal{Q}X \perp \mathcal{Q}Y$, we have $U_1 \perp U_2$. Viceversa let suppose that I is a complex structure and (I, J, K) an adapted basis. Let $I' = \alpha I + \beta J + \gamma K$, with $\alpha^2 + \beta^2 + \gamma^2 = 1$. Then $U_1 \perp U_2$ if

$$\begin{aligned}
&0 =< Y, X > \\
&0 =< Y, IX > \\
&0 =< I'Y, X > =< \alpha IY + \beta JY + \gamma KY, X > \Rightarrow -\beta < Y, JX > -\gamma < Y, KX > = 0 \\
&0 =< I'Y, IX > =< \alpha IY + \beta JY + \gamma KY, IX > \Rightarrow -\gamma < Y, JX > +\beta < Y, KX > = 0,
\end{aligned}$$

This implies

$$\begin{cases} Y \perp U_1 \\ (-\beta^2 - \gamma^2) < Y, JX > = 0. \end{cases}$$

Then

$$U_2 \perp U_1 \Leftrightarrow \begin{cases} I' = \pm I \ \text{ and } \ Y \perp U_1 & \text{or} \\ Y \perp \mathcal{Q}X \end{cases} \tag{5}$$

since from above $< Y, JX > = 0 \Rightarrow < Y, KX > = 0$.

Then a pair of complex 2-planes $U_1 = L(X, IX)$, $U_2 = L(Y, I'Y)$ are orthogonal iff whether $I = \pm I'$ and $Y \perp U_1$ (but not necessarily to $\mathcal{Q}X$) or for the quaternionic subspaces $\mathcal{Q}X \perp \mathcal{Q}Y$. [2] □

[2] This Lemma is true also in a para-quaternionic Hermitian vector space. In that case the Lemma applies not only to a pair of 2-dimensional complex subspaces but also to a pair of 2-dimensional para-complex subspaces or to a pair made of a 2-dimensional complex and a 2-dimensional para-complex subspace. For interested readers, in [16] and [18] it is possible to find the analogue decomposition of a para-quaternionic Hermitian vector space which differs from (4) because of the existence, in that case, of (weakly) para-complex and nilpotent subspaces.

It follows that if $Y \in \mathcal{Q}X$ then $L(Y, \tilde{I}Y)$ is never orthogonal to U_1 unless $\tilde{I} = \pm I$ (in which case $L(Y, \pm IY) = L(JX, KX)$, i.e. a pair of complex 2-planes by different (up to sign) complex structures and belonging to the same quaternionic line are never orthogonal to each other. Viceversa, since any 2-plane in a quaternionic line is $\tilde{I}$-complex for some $\tilde{I} \in S(\mathcal{Q})$, if U_1 and U_2 are orthogonal 2-planes belonging to the same quaternionic line then they are complex by the same complex structure. We can now state the

Proposition 1.11. *Strictly orthogonal complex subspaces by different complex structures are orthogonal in Hermitian sense i.e. their quaternionifications are strictly orthogonal in pair. In particular the different complex addends in U^{Σ} given in (4) are orthogonal in Hermitian sense.*

Proof. Let consider the pair (U_1, I_1) and (U_2, I_2), $I_1 \neq \pm I_2$, of strictly orthogonal complex subspaces. Any I-complex subspace can be decomposed into the orthogonal sum of I-complex 2 planes. From (1.10), any 2-plane of the decomposition of (U_1, I_1) is Hermitian orthogonal to any 2-plane of the decomposition of (U_2, I_2) then ${U_1}^{\mathbb{H}} \perp {U_2}^{\mathbb{H}}$. We conclude then the different addends of U^{Σ} belong to strictly orthogonal quaternionic subspaces. $\square$

Definition 1.12. *We call* Σ***-complex subspace*** *a pure subspace $U \subset (V^{4n}, \mathcal{Q}, <,>)$ orthogonal sum of maximal complex subspaces (U_i, I_i) with $I_i \in S(\mathcal{Q})$. We denote it by $(U, \mathcal{I})$ where $\mathcal{I} = \{I_i\}$.*

Fixed an admissible basis (I, J, K), we need to order the set $\mathcal{I} = \{I_i = \alpha_i I + \beta_i J + \gamma_i K\}$. A way to do it is for instance by using the lexicographic order of the coefficients $\alpha_i, \beta_i, \gamma_i$. By Proposition (1.11), the complex addends of a Σ-complex subspace are orthogonal in Hermitian sense.

To prove the following proposition concerning the decomposition of a Σ-complex subspace, we need the

Lemma 1.13. *The orthogonal sum of a pair of pure complex subspaces (U_1, I), (U_2, I'), $I \neq \pm I'$ contains no other complex subspace (not contained in U_1 or U_2).*

Proof. Let (U_1, I) and (U_2, I'), $I \neq \pm I'$ be a pair of orthogonal pure complex subspaces and consider their sum $U = U_1 \overset{\perp}{\oplus} U_2$. From Proposition (1.11) one has that $U_1^{\mathbb{H}} \perp U_2^{\mathbb{H}}$. Suppose there exists $0 \neq T \in U$, $T = X + Y$, $X \in U_1, Y \in U_2$ and $\widetilde{I} = \alpha I + \beta J + \gamma K \in S(\mathcal{Q})$ such that $\widetilde{I}T \in U$. The vector $\widetilde{I}T = (\alpha IX + \beta JX + \gamma KX) + (\alpha IY + \beta JY + \gamma KY)$ is orthogonal sum (in a unique way) of a vector $T_1 \in U_1$ and a vector $T_2 \in U_2$. The vector $\alpha IX \in U_1$ whereas the vector $\beta JX + \gamma KX \in U_1^{\perp}$ and also $\beta JX + \gamma KX \perp (\alpha IY + \beta JY + \gamma KY) \in QY$. Then necessarily $\beta JX + \gamma KX = 0$ i.e. $I(JX) = \frac{-\beta}{\gamma} JX$. Since I has no real eigenvalues, such equation is satisfied if $X = 0$ or if $\beta = \gamma = 0$ i.e. if $\widetilde{I} = \pm I$. Analogously, if $\widetilde{I} = \alpha' I' + \beta' J' + \gamma' K'$ one has that $\beta' J'Y + \gamma' K'Y = 0$ which implies that $Y = 0$ or $\beta' = \gamma' = 0$ i.e. if $\widetilde{I} = \pm I'$. Then excluding the case $X = Y = 0$ the only possibilities are $\widetilde{I} = \pm I$ and $Y = 0$ or $\widetilde{I} = \pm I'$ and $X = 0$. The first implies that $T = X \in U_1$ and $\tilde{T} = IX \in U_1$; the second implies $T = Y \in U_2$ and $\tilde{T} = I'Y \in U_2$. $\square$

Extending the previous proof to a finite number of addends we can then state the following

Proposition 1.14. *Let $U \subset V$ be a Σ-complex subspace. Then all other subspaces in U (not contained in the complex addends) are totally real. In other words the given decomposition of U into an Hermitian orthogonal sum of maximal pure complex subspaces by different structure (up to sign) is unique.*

This results applies in particular to the Σ-complex subspace U^Σ of the decomposition given in (4).

We finally underline that in general *the Euclidean orthogonal sum $U^\Sigma \overset{\perp}{\oplus} U_R$ in (1.8) is not orthogonal in Hermitian sense.* Moreover the two addends are not canonically defined as can be easily seen by considering for example the 4-dimensional pure subspace $U = (U_1, I_1) \oplus (U_2, I_2),\ I_1 \neq I_2$ direct but not orthogonal sum of a pair of (totally) complex 2-planes. The decomposition $U = U_1 \overset{\perp}{\oplus} (U_1^\perp \cap U)$ and $U_2 \overset{\perp}{\oplus} (U_2^\perp \cap U)$ are two different orthogonal decomposition where $U_1^\perp \cap U$ and $U_2^\perp \cap U$ are different totally real 2-planes.

2 Preliminaries

We define the (Euclidean) angle between two subspaces of dimension p and q of an Euclidean vector space E^n by using exterior algebra (see [14] among others). Let $(E^n, <, >)$ be an n-dimensional vector space endowed with an Euclidean scalar product. Any decomposable p-vector $\alpha = a_1 \wedge \ldots \wedge a_p \in \Lambda^p E^n$ corresponds to an oriented subspace $A^p \in E^n$ and precisely to the one spanned by $a_1, \ldots, a_p$. Conversely, for any basis of A^p the wedge of these vectors is a multiple of α (i.e. it is equal to $k\alpha$, with $k \in \mathbb{R}, k \neq 0$). The scalar product $<, >$ in E^n induces a scalar product $\cdot$ in the vector space $\Lambda^p E^n$ by defining

$$\alpha \cdot \beta = det(< a_i, b_j >)$$

for a pair of decomposable vectors $\alpha = a_1 \wedge \ldots, \wedge a_p; \qquad \beta = b_1 \wedge \ldots, \wedge b_p, \quad a_i, b_i \in E^n$ and then extending for linearity to any pair of vectors of $\Lambda^p E^n$.

It is definite positive and non degenerate then the pair $(\Lambda^p E^n, <>)$ is an Euclidean vector space. In particular for the angle between α and β,

$$\cos \widehat{\alpha\beta} = \frac{\alpha \cdot \beta}{\sqrt{\alpha \cdot \alpha}\,\sqrt{\beta \cdot \beta}} = \frac{\det(< a_i, b_i >)}{mis\ \alpha\ mis\ \beta}. \tag{6}$$

being

$$mis\ \alpha = |\alpha| = \sqrt{\alpha \cdot \alpha}.$$

Given A^p and B^q and $\alpha = a_1 \wedge \ldots, \wedge a_p \in \Lambda^p E^n$ associated to A and $\beta = b_1 \wedge \ldots, \wedge b_q \in \Lambda^q E^n$ associated to B, we consider the orthogonal projections of $a_1, \ldots, a_p$ on B and $B^\perp$. Then $a_i = a_i^H + a_i^V$, and $\alpha = \alpha_H + \alpha_V + \alpha_M$ (where M stands for mixed part).

If we choose another basis in A (then $\alpha' = k\alpha$) we have

$$\alpha'_H = k\alpha_H, \quad \alpha'_V = k\alpha_V, \quad \alpha'_M = k\alpha_M.$$

Definition 2.1. *The angle $\widehat{A,B}$ between the non oriented subspaces A^p and B^q, $p \leq q$ is the usual angle (between two lines, a line and a plane, two planes) i.e. the angle between one subspace and its orthogonal projection onto the other i.e.*

$$\theta = \arccos \frac{|\alpha_H|}{|\alpha|}.$$

Then $\theta \in [0, \pi/2]$ and, from previous Lemma, it is independent from the chosen basis in A. In particular, if $p = q$ we can write

$$\theta = \arccos \frac{|\det(< a_i, b_j >)|}{|\alpha|\ |\beta|} \tag{7}$$

i.e. *the cosine of the angle between a pair of p-planes $A, B \subset E^n$ equals the absolute value of the cosine of the angle between any pair of p-vectors $\alpha, \beta \in \Lambda^p E^n$ corresponding to A and B.*

Remark 2.2. *In case we consider oriented subspaces of E^n then we do not take the absolute value in the previous expressions and one has $\theta \in [0, \pi]$. Unless expressively stated, no orientation will be defined on the subspaces we consider in this paper.*

We recall the definition of the principal angles between a pair of subspaces of a real vector space V (see [6],[10] among others).

Definition 2.3. *Let $A, B \subseteq V$ be subspaces, $\dim k = dim(A) \leq dim(B) = l \geq 1$. The **principal angles** $\theta_i \in [0.\pi/2]$ between the subspaces A and B are recursively defined for $i = 1, \ldots, k$ by*

$$\cos\theta_i =< a_i, b_i >=$$

$$\max\{< a, b >: ||a|| = ||b|| = 1,\ a \perp a_m,\ b \perp b_m,\ m = 1, 2, \ldots, i-1\}.$$

*The unitary vectors $\{a_j\}, \{b_j\}, j = 1, \ldots, k$ are called the **principal vectors** of the pair (A, B), in particular $(a_j, b_j) \in (A \times B), j = 1, \ldots, k$ are <u>related</u> principal vectors corresponding to θ_j.*

In words, the procedure is to find the unit vector $a_1 \in A$ and the unit vector $b_1 \in B$ which minimize the angle between them and call this angle θ_1. Then consider the orthogonal complement in A to a_1 and the orthogonal complement in B to b_1 and iterate. The principal angles $\theta_1, \ldots, \theta_k$ between the pair of subspaces A, B are some of the critical values of the angular function

$$\phi_{A,B} = A \times B \to \mathbb{R}$$

associating with each pair of non-zero vectors $a \in A,\ b \in B$ the angle between them. In the following, given a pair of subspaces A, B we will denote by $Pr_A^B : A \to B$ (resp. Pr_B^A) the orthogonal projector of A onto B (resp. B onto A). The principal angles are the diagonal entries of the orthogonal projector Pr_B^A stated in the theorem of Afriat ([8], [1]):

Theorem 2.4. *[1], [8]. In any pair of subspaces A^k and B^l there exist orthonormal bases $\{u_i\}_{i=1}^k$ and $\{v_j\}_{j=1}^l$ such that $< u_i, v_i >\geq 0$ and $< v_i, v_j >= 0$ if $i \neq j$.*

Proof. It is a direct consequence of the following

Lemma 2.5. *Given finite dimensional subspaces A, B, let a_1, b_1 attain*

$$\max\{< a, b >, \quad a \in A,\ b \in B, \quad ||a|| = 1,\ ||b|| = 1\}$$

(i.e. the pair (a_1, b_1) are the first pair of related principal vectors). Then

1. *for some $\alpha \geq 0$,*
$$Pr_A^B a_1 = \alpha b_1, \qquad Pr_B^A b_1 = \alpha a_1$$

2. *$a_1 \perp (b_1^\perp \cap B)$ and $b_1 \perp (a_1^\perp \cap A)$ which leads the diagonal form of the matrix of the Projector Pr_A^B (and Pr_B^A).*

To see that 1) holds, note that $Pr_A^B a_1 = \alpha b$ where α, b minimize $||a_1 - \alpha b||^2$ for $b \in B, ||b|| = 1$ and α a scalar. Thus to minimize $||a_1 - \alpha b||^2 = \alpha^2 - 2\alpha < a_1, b > +1$ we must maximize $< a_1, b >$. Moreover $\alpha =< a_1, b_1 >$ is the cosine of the first principal angle.

For 2), let $A_1 = a_1^\perp \cap A$ (resp. $B_1 = b_1^\perp \cap B$). If $a \in A_1$, then $a \perp b_1$ since $< a, b_1 >=< Pr_A^A a, b_1 >=< a, P_A^A b_1 >=< a, \alpha a_1 >= 0$. Likewise if $b \in B_1$ then $b \perp a_1$. We proceed letting a_2 and b_2 attain

$$\max\{< a, b >, \quad a \in A_1,\ b \in B_1, \quad ||a|| = 1,\ ||b|| = 1\}$$

and continue till we have exhausted A and B. □

From (1) of (2.5), one has that the cosines of the principal angles between the pair of subspaces A, B of V can also defined as the singular values of the orthogonal projector Pr_B^A (or equivalently Pr_A^B). If α is a singular value, we call the pair $(a, b) \in (A \times B)$ such that $Pr_A^B a = \alpha b, \qquad Pr_B^A b = \alpha a$ *related* singular vectors (associated to α).

We underline the following relation between the angle and the principal angles between a pair of subspaces of a real vector space V (see [19]).

Proposition 2.6. *[13] Let A^p and B^q be a pair of subspaces of V^n with $1 \leq p \leq q \leq n$. Let θ be the angle between the subspaces A^p and B^q and $\theta_1, \ldots, \theta_p$ the set of principal angles. Then*

$$\cos\theta = \cos\theta_1 \cdot \cos\theta_2 \cdot \ldots \cdot \cos\theta_p.$$

In particular, if $p = q$, one has the well known result $|\det(G)| = \cos\theta_1 \cdot \cos\theta_2 \cdot \ldots \cdot \cos\theta_p$ where G is the matrix representing the Projector Pr_A^B w.r.t. some orthonormal pair of bases.

Let recall the definition and some properties of isoclinic subspaces.

Definition 2.7. *A pair of non oriented subspaces A and B of same dimension are said to be **isoclinic** and the angle ϕ $(0 \leq \phi \leq \frac{\pi}{2})$ is said to be angle of isoclinicity between them if either of the following conditions hold:*
1) the angle between any non-zero vector of one of the subspaces and the other subspace is equal to ϕ;
2) $GG^t = \cos^2\phi\, Id$ for the matrix $G =< a_i, b_j >$ of the orthogonal projector $P_B^A : B \to A$ with respect to any orthonormal basis $\{a_i\}$ of A and $\{b_j\}$ of B;
3) all principal angles between A and B equal ϕ.

Definition 2.8. *We denote by $\mathcal{IC}^{2m}$ the set of $2m$-dimensional subspaces of V such that, for any $A \in S(\mathcal{Q})$, the pair (U, AU) is isoclinic. When we do not need to specify the dimension we just use the notation $\mathcal{IC}$ and we call them simply **isoclinic subspaces**.*

The fact that we consider only even dimensions subspaces follows from the

Proposition 2.9. *Let U be an odd dimension isoclinic subspace. Then U is a real hermitian product subspace (r.h.p.s.). Namely $\mathcal{IC}^{2m+1}$ is the set of all and only the real Hermitian product $(2m+1)$-dimension subspaces.*

Proof. If U is a r.h.p.s. then by definition it is an isoclinic subspace. Viceversa U is isoclinic and $\dim U = 2m+1$, for any $A \in S(\mathcal{Q})$, by the skew-symmetry of ω^A one (and then all) principal angle is necessarily equal to $\pi/2$.

□

Fixed an admissible basis (I, J, K) and, given $U \in \mathcal{IC}^{2m}$, we denote by $\theta^I, \theta^J, \theta^K$ the respective angles of isoclinicity. In [21] we introduced the following definitions. If the pair (U, IU) (resp. (U, JU), resp. (U, KU)) is strictly orthogonal (i.e. if all principal angles are $\pi/2$) we said that U is I-orthogonal (resp. J-orthogonal, resp. K-orthogonal) and in general we spoke of single orthogonality (or 1-orthogonality). When two (resp. three) of the above pair are strictly orthogonal we spoke of double (resp. triple)-orthogonality. By saying that U is **orthogonal** (without specifying the complex structures) we mean that at least one among $\theta^I, \theta^J, \theta^K$ equals $\pi/2$. Observe that only r.h.p. subspaces have a triple orthogonality. In particular in this paper, the isoclinic subspaces we consider have no orthogonality unless they are totally complex in which case they have a double orthogonality.

Let U be a subspace. Let fix an admissible basis $\mathcal{H}$ of $\mathcal{Q}$, and let $f : U \times U \times \ldots \times U \to W$ some function where the codomain W is some vector space. If f is constant on its domain, we will say that f is an *invariant* of U. If furthermore the invariant f does not depend on the chosen hypercomplex basis $\mathcal{H}$, we will say that f is an *intrinsic property* of U.

Finally we recall the notion of Kähler angle which is defined in a real vector space V endowed with a complex structure I.

Definition 2.10. *Let (V^{2n}, I) be a real vector space endowed with a complex structure I. For any pairs of non parallel vectors $X, Y \in V$ their **Kähler angle** is given by*

$$\Theta^I = \arccos \frac{< X, IY >}{|X|\,|Y| \sin \widehat{XY}} = \arccos \frac{< X, IY >}{mis\ (X \wedge Y)}. \tag{8}$$

Then $0 \leq \Theta^I \leq \pi$. It is straightforward to check that the Kähler angle is an intrinsic property of the oriented 2-plane $U = L(X,Y)$. For this reason one speaks of the Kähler angle of a 2-plane. The Kähler angle measures the deviation of a 2-plane from holomorphicity. For instance the Kähler angle of a r.h.p. subspace U is $\Theta^I(U) = \pi/2$ and the one of a complex plane (U, I) is $\Theta^I(U) \in \{0, \pi\}$.

The cosine of the Kähler angle of the pair of 2-planes with opposite orientation U and $\tilde{U} = L(Y,X)$ have opposite sign i.e. $\cos\Theta^I(U) = -\cos\Theta^I(\tilde{U})$, then, if one disregards the orientation of the 2-plane U, we can consider the absolute value of the right hand side of equation (8) restricting the Kähler angle to the interval $[0, \pi/2]$. In this case the Kähler angle of the 2-plane U coincides with one of the two identical principal angles, say $\theta^I(U)$, between the pairs of 2-plane U and IU (same as the ones of the pair $(\tilde{U}, I\tilde{U})$) which are always isoclinic as one can immediately verify, then

$$\cos\theta^I(U) = |\cos\Theta^I(U)|$$

and from (2.6), one has

$$\cos(\widehat{U, IU}) = \cos(\widehat{\tilde{U}, I\tilde{U}}) = \cos^2\theta^I(U) = \cos^2\Theta^I(U) = \frac{< X, IY >^2}{mis^2\ (X \wedge Y)}. \tag{9}$$

Generalizing the notion of Kähler angle, in an Hermitian quaternionic vector space $(V^{4n}, \mathcal{Q}, <,>)$ we will speak of the **A-Kähler angle** of an oriented 2-plane U with $A \in S(\mathcal{Q})$ and denote it by $\Theta^A(U)$.

In [21] we proved the following

Proposition 2.11. *Let $U \subset (V^{4n}, \mathcal{Q}, <,>)$ be a 2 plane. The sum of the cosines of the angles between the pairs (U, IU), (U, JU), (U, KU) is constant for any admissible basis (I, J, K) of $\mathcal{Q}$.*

It follows that the quantity

$$\cos^2\Theta^I(U) + \cos^2\Theta^J(U) + \cos^2\Theta^K(U) = \cos^2\theta^I(U) + \cos^2\theta^J(U) + \cos^2\theta^K(U) \tag{10}$$

is an intrinsic property of a 2-plane.

Given a subspace U of an Hermitian quaternionic vector space and generalizing a well known notion relative to an Hermitian complex vector space, for any $A \in S(\mathcal{Q})$ we will call **A-Kähler form** of U the skew-symmetric bilinear form

$$\begin{array}{rccl} \omega^A|_U : & U \times U & \to & \mathbb{R} \\ & (X,Y) & \mapsto & < X, AY > . \end{array}$$

It is well known that the A-Kähler form admits a *standard form*, namely w.r.t. an orthonormal *standard basis* $(X_1, \ldots, X_m)$ of the m-dimensional subspace U one has

$$(\omega^A_{ij}) = (< X_i, AX_j >) = \begin{cases} \geq 0 & \text{if } i \text{ is odd and } j = i+1, \\ 0 & \text{otherwise}, \end{cases}$$

for $i \leq j \leq k$.

A standard form determines some ω^A-invariant subspaces U_i^A.[3] Such subspaces are uniquely defined whereas the standard bases of $\omega^A|_{U_i^A}$ are not: namely, given one of them, all others are obtained through an orthogonal transformations of U_i^A commuting with the complex structure $A \in S(\mathcal{Q})$). Observe that all such bases have the same orientation.

The following definitions are specific of an Hermitian quaternionic vector space.

Definition 2.12. *[3] The* ***characteristic angle*** $\theta \in [0, \frac{\pi}{2}]$ *of a pair of vector* L, M *is the angle of the 1-dimensional quaternionic subspace they generate i.e.* $\theta = \widehat{\mathcal{Q}L, \mathcal{Q}M}$ *where* $\mathcal{Q}L = L(L, IL, JL, KL)$, $\mathcal{Q}M = L(M, IM, JM, KM)$ *and* (I, J, K) *is any admissible basis. It is given by*

$$\cos\theta(L, M) = \frac{[\mathcal{N}(L \cdot M)]^2}{misL^4\ misM^4} =$$

$$\frac{[< L, M >^2 + < L, IM >^2 + < L, JM >^2 + < L, KM >^2]^2}{mis^4 L\ mis^4 M}$$

The ***Hermitian angle*** θ_h *of a pair of vector* L, M *is*

$$\cos\theta_h(L, M) = \frac{|L \cdot M|}{misL\ misM} =$$

$$\frac{\sqrt{< L, M >^2 + < L, IM >^2 + < L, JM >^2 + < L, KM >^2}}{misL\ misM}.$$

Then

$$\cos\theta(L, M) = \cos^4\theta_h(L, M)$$

We conclude this preliminary section recalling that the group $Sp(1)$ is the group under multiplication of unitary quaternions. It is a Lie group whose Lie algebra $\mathfrak{sp}_1 = Im\,\mathbb{H}$ is a quaternionic structure on $\mathbb{H}^n$.

Let $(V^{4n}, \mathcal{Q}, <, >)$ be an Hermitian quaternionic vector space. For any quaternion $q \in Sp(1)$, let consider the unitary homothety in the $\mathbb{H}$-module $V \simeq \mathbb{R}^{4n}$.

$$R_q : X \mapsto Xq, \quad X \in V.$$

To these transformations belong for instance the automorphisms $I = R_{-i}, J = R_{-j}, K = R_{-k}$.

Proposition 2.13. *[4] The unitary homotheties are rotations of* V^{4n} *that preserve any quaternionic line. Moreover for any* $X \in V$ *the angle* $\widehat{X, Xq}$ *does not depend on* X *and it is*

$$\cos\widehat{X, Xq} = Re(q)$$

The action of $Sp(1)$ determines then an inclusion

$$\begin{array}{rccc} \lambda: & Sp(1) & \hookrightarrow & SO(4n) \\ & q & \mapsto & R_q \end{array} \tag{11}$$

[3] Using the same notation used in [20] we call ω^A-invariant subspaces the T-invariant subspaces of the endomorphism T of U represented by the same matrix of ω^A.

We define $Sp(n)$ to be the subgroup of $SO(4n)$ commuting with $\lambda(Sp(1))$ i.e. $Sp(n)$ is the centralizer of $\lambda Sp(1)$ in $SO(4n)$.

For completeness we recall the following definition. Let consider in $\mathbb{H}$-module V the transformations $T_{(A,q)} : X \mapsto AXq$ with $A \in Sp(n),\ q \in Sp(1),\ X \in V$. We denote by $Sp(n) \cdot Sp(1)$ the group of these transformations.

The group $Sp(n) \cdot Sp(1)$ is the normalizer of $\lambda Sp(1)$ in $SO(4n)$ which is isomorphic to the quotient $Sp(n) \times_{\mathbb{Z}_2} Sp(1)$ where $\mathbb{Z}_2 = \{1, -1\}$. Note that $Sp(1) \cdot Sp(1)$ is precisely $SO(4)$, whereas for $n \geq 2$ $Sp(n) \cdot Sp(1)$ is a maximal Lie subgroup of $SO(4n)$. Observe that $Sp(n) \cdot Sp(1)$ is not a subgroup of $U(2n)$. For a deeper understanding of the groups $Sp(n)$ and $Sp(n) \cdot Sp(1)$ one can refer among others to [12] and [7].

3 $Sp(n)$-orbits of complex and Σ-complex subspaces in the real Grassmannians

3.1 Theorems about the $Sp(n)$-orbits of a generic subspace

Let $(V^{4n}, \mathcal{Q}, <,>))$ be an Hermitian quaternionic vector space. Recalling the expression of the Hermitian product given in (3), in [20] we find the following characterizations of the $Sp(n)$-orbits in the real Grassmannians $Gr^{\mathbb{R}}(m, 4n)$.

Theorem 3.1. *[20] Let U and W be a pair of subspaces of real dimension m in the $\mathbb{H}$-module V^{4n}. Then there exists $A \in Sp(n)$ such that $AU = W$ iff there exist bases $\mathcal{B}_U = (X_1, \dots, X_m)$ and $\mathcal{B}_W = (Y_1, \dots, Y_m)$ of U and W respectively w.r.t. which for the Hermitian products one has $(X_i \cdot X_j) = (Y_i \cdot Y_j),\ i = 1, \dots, m$ for one and hence any admissible basis of $\mathcal{Q}$.*

Let $U \subseteq V$ be a subspace. For any $A \in S(\mathcal{Q})$ we denoted by $\mathcal{B}^A(U)$ the set of the standard bases of $\omega^A|_U$ and by $\boldsymbol{\theta}^A(U)$ the principal angles between the pair (U, AU). Moreover, fixed an admissible basis (I, J, K), $\mathcal{B}(U) = \{B_I, B_J, B_K\}$ is the set made of triples of bases of U with $B_I \in \mathcal{B}^I(U), B_J \in \mathcal{B}^J(U), B_K \in \mathcal{B}^K(U)$.

Necessary and sufficient conditions for the pair U, W to belong to the same $Sp(n)$-orbit in the real Grassmannian are also stated in

Theorem 3.2. *[20] Let (I, J, K) be an admissible basis of $\mathcal{Q}$. The subspaces U^m and W^m of V are in the same $Sp(n)$-orbit iff*

1. *they share the same I, J, K-principal angles i.e.*

$$\boldsymbol{\theta}^I(U) = \boldsymbol{\theta}^I(W), \quad \boldsymbol{\theta}^J(U) = \boldsymbol{\theta}^J(W), \quad \boldsymbol{\theta}^K(U) = \boldsymbol{\theta}^K(W)$$

   for one and hence any hypercomplex basis (I, J, K) or equivalently the singular values of the projectors $Pr^{IU}, Pr^{JU}, Pr^{KU}$ equals those of $Pr^{IW}, Pr^{JW}, Pr^{KW}$ for one and hence any admissible basis (I, J, K);

2. *there exist $(\{\tilde{X}_i\}, \{\tilde{Y}_i\}, \{\tilde{Z}_i\}) \in \mathcal{B}(U)$ and $(\{\tilde{X}'_i\}, \{\tilde{Y}'_i\}, \{\tilde{Z}'_i\}) \in \mathcal{B}(W)$ such that $A = A',\ B = B'$ where*

$$A = (< \tilde{X}_i, \tilde{Y}_j >), \qquad A' = (< \tilde{X}'_i, \tilde{Y}'_j >), \qquad B = (< \tilde{X}_i, \tilde{Z}_j >),$$

$$B' = (< \tilde{X}'_i, \tilde{Z}'_j >).$$

The determination of the principal angles between a pair of subspaces S, T is a well know problem solved by the singular value decomposition of the orthogonal projector of S onto T. Here, for a chosen $U \subset V$ we consider the pairs (U, AU), $A \in S(\mathcal{Q})$ and denote by Pr^{AU} the orthogonal projector of U onto AU. In this case the singular values of Pr^{AU} are always degenerate which implies that they have non-unique singular vectors. In terms of the principal vectors of the pair (U, AU) we can equivalently say that the principal vectors are never uniquely defined not even if U is 2-dimensional. We do not consider the ambiguity in sign of the principal vectors which is always existing also in dimension one.

Another way to obtain the principal angles and the associated principal vectors between the pair of subspaces (U, AU), for any $A \in S(\mathcal{Q})$, is through the standard decompositions of the restriction to U of the A-Kähler skew-symmetric form ω^A : $(X, Y) \mapsto < X, AY >, \; X, Y \in U$.

Recalling that a *standard basis* is an orthonormal basis w.r.t. which $\omega^A|_U$ assumes standard form, in the following we will consider it ordered according to non increasing value of the non-negative entries. Denoting by $\mathcal{B}^A(U)$ the set of all such bases, one has that any $B \in \mathcal{B}^A(U)$ consists of the principal vectors of the pair (U, AU) (ordered as just explained).

The problem to determine the $Sp(n)$-orbits turns therefore into the one of determining the existence of such triple of bases. In [20] we set up a procedure to determine a triple of *canonical bases* w.r.t. which one computes the matrices A and B. Namely, fixed an admissible basis (I, J, K), a triple of canonical bases of a subspace U is constituted by a triple of standard bases of $\omega^I, \omega^J, \omega^K$ that either are uniquely determined by the procedure aforementioned or, if this is not the case, nevertheless the associated matrices A, B are unique. We call them *canonical matrices* and denote by C_{IJ} and C_{IK}. If the pair U, W belong the same $Sp(n)$-orbit, the action of the group maps the canonical basis of U onto the ones of W. Therefore the determination of such standard bases let us restrict the statement of Theorem (3.2) to the canonical matrices leading to the

Theorem 3.3. *[20] Let (I, J, K) be an admissible basis of $\mathcal{Q}$. The subspaces U and W of V are in the same $Sp(n)$-orbit iff $Inv(U) = Inv(W)$ where*

$$Inv(U) = \{\boldsymbol{\theta}^I(U), \, \boldsymbol{\theta}^J(U), \, \boldsymbol{\theta}^K(U), C_{IJ}, \; C_{IK}\}.$$

It is straightforward to verify that if $Inv(U) = Inv(W)$ w.r.t. the admissible basis (I, J, K) then $Inv(U) = Inv(W)$ for any admissible basis.

3.2 The 2-dimensional complex subspace

Let $(V^{4n}, \mathcal{Q}, <,>)$ be an Hermitian quaternionic vector space. To study the $Sp(n)$-orbits of a $2m$-dimensional complex and Σ-complex subspace of V in the real Grassmannian $Gr^{\mathbb{R}}(2m, 4n)$ we will need the theory of the isoclinic subspaces developed in [21]. In particular here we use some of the results obtained regarding the 4-dimensional case as we will see that complex and Σ-complex subspaces admit an orthogonal decomposition into 4-dimensional isoclinic addends (plus a totally complex 2-dimensional subspace if m is odd). Furthermore such addends will be complex by some compatible complex structure and Hermitian orthogonal in pairs.

In the following then we recall some results about isoclinic subspaces referring for a more extensive analysis to [21].

Let initiate our study by considering a 2-dimensional I-complex subspace. By the skew-symmetry of the A-Kähler form for any $A \in S(\mathcal{Q})$, any 2-dimensional subspace of U is isoclinic with AU. Therefore as a set one has that $G_{\mathbb{R}}(2, 4n) = \mathcal{IC}^2$.

In particular all 2-dimensional complex subspaces are totally complex. The invariants characterizing the $Sp(n)$-orbits in the Grassmannian of 2-planes are determined in [19] where we also studied the analogue problem for the group $Sp(n) \cdot Sp(1)$.

Fixed an admissible basis (I, J, K), let consider an oriented 2-dimensional subspace $U \subset V$ generated by the oriented basis (M, L). In [19] we introduced the purely imaginary quaternion

$$\mathcal{IM}(U) = \frac{Im(L \cdot M)}{mis(L \wedge M)}, \quad L, M \in U. \tag{12}$$

We showed that it is an intrinsic property of an oriented 2-plane $U \subset (V^{4n}, \mathcal{Q}, <,>)$ i.e. it does not depend neither on the chosen generators L, M nor on the admissible basis $\mathcal{H}$ of $\mathcal{Q}$. Moreover $Sp(n)$ preserves $\mathcal{IM}(U)$. In particular, if the pair L, M is an orthonormal oriented basis of U, then $\mathcal{IM}(U) = L \cdot M$. We called it **Imaginary measure** of the 2-plane U. Disregarding the orientation of U and being (L, M) some orthonormal basis, it is $\mathcal{IM}(U) = \{\pm L \cdot M\}$ i.e. it is the set made of a pair of conjugated pure imaginary quaternions. We proved that

Theorem 3.4. *[19] The imaginary measure $\mathcal{IM}(U)$ represents the full system of invariants for the $Sp(n)$-orbits in the real Grassmannian of 2-planes $G_{\mathbb{R}}(2, 4n)$ as well as in $G^+_{\mathbb{R}}(2, 4n)$ (the Grassmannian of the oriented 2-planes) i.e. a pair of 2-planes (U, W) of $(V^{4n}, \mathcal{Q}, <,>)$ are in the same $Sp(n)$-orbit iff $\mathcal{IM}(U) = \mathcal{IM}(W)$.*

Let consider a triple of standard bases $(X_1, X_2), (X_1, Y_2), (X_1, Z_2)$ with a common leading vector X_1 of the non oriented 2-plane U. By definition one has that $\cos\theta^I = < X_1, IX_2 >, \cos\theta^J = < X_1, JY_2 >, \cos\theta^K = < X_1, KZ_2 >$ are non negative, and computed $\xi = < X_2, Y_2 >, \chi = < X_2, Z_2 >, \eta = < Y_2, Z_2 >$, where $(\xi, \chi, \eta) \in \{-1, 1\}$ one has that the matrices C_{IJ} and C_{IK} w.r.t. the standard bases $(X_1, X_2), (X_1, Y_2)(X_1, Z_2)$ are given by

$$C_{IJ} : \begin{pmatrix} 1 & 0 \\ 0 & \xi \end{pmatrix} \qquad C_{IK} : \begin{pmatrix} 1 & 0 \\ 0 & \chi \end{pmatrix} \tag{13}$$

It is straightforward to verify that the pair (ξ, χ) is an invariant of U. Then any triple of standard bases of $\omega^I, \omega^J, \omega^k$ with a common leading vector are canonical bases of U whose canonical matrices are given in (13). Therefore, according to Theorem (3.2), the pair (ξ, χ) together with the triple $(\theta^I, \theta^J, \theta^K)$, determines the $Sp(n)$-orbits of the (non oriented) 2-plane U.

This is accordance with the Theorem (3.4). In fact, If U has a triple orthogonality then clearly $\mathcal{IM}(U) = 0$. In this case any orthonormal basis is at the same time a standard basis of $\omega^I, \omega^J, \omega^K$ which implies $\xi = \chi = 1$. Else, suppose without lack of generality that $\cos\theta^I \neq 0$ and let (X_1, X_2) be an ω^I-standard basis. Then

$$\mathcal{IM}(U) = X_1 \cdot X_2 = \pm(\cos\theta^I i + \xi \cos\theta^J j + \chi \cos\theta^K k).$$

Given a pair of 2-planes U, W with $\mathcal{IM}(U) = \mathcal{IM}(W)$, according to Theorem (3.4), they are in the same orbit. Since they share the same pair (ξ, η) and the same triple $(\theta^I, \theta^J, \theta^K)$, they are in the same $Sp(n)$-orbit also according to Theorem (3.2). Viceversa if they share the same pair (ξ, η) and the same triple $(\theta^I, \theta^J, \theta^K)$ which implies that they belong to the same $Sp(n)$-orbit according to Theorem (3.2) then clearly $\mathcal{IM}(U) = \mathcal{IM}(W)$.

A 2-dimensional I-complex subspace is totally complex being clearly $U \perp JU = KU$. Then for any admissible basis (I, J, K) one has $(\theta^I, \theta^J, \theta^K) = (1, 0, 0)$. Furthermore for $X \in U$ the ω^I-standard basis $(X, -IX)$ can be considered a standard basis of $\omega^{K'}$ centered on X for any $K' \in I^\perp$. Then we assume $\xi = \chi = 1$ and w.r.t. any standard and canonical ω^I-basis, for the canonical matrices one has $C_{IJ} = C_{IK} = Id$. Then we can conclude affirming

Theorem 3.5. *All and only the I-complex 2-dimensional subspaces of $(V^{4n}, \mathcal{Q}, <,>)$ belong to one $Sp(n)$-orbit in $Gr^{\mathbb{R}}(2, 4n)$ or equivalently $Sp(n)$ is transitive on the set of the 2-dimensional I-complex subspaces of V.*

We obtain the same result applying the Theorem (3.4) since if U is an I-complex 2-plane then $\mathcal{IM}(U) = \pm i$.

Furthermore, w.r.t. any ω^I-standard basis, the matrix of the Hermitian product is given by

$$H_{\mathcal{B}}(U) = \begin{pmatrix} 0 & i \\ & \\ -i & 0 \end{pmatrix}. \tag{14}$$

Then again the same results follows from Theorem (3.1).

For completeness we report the corresponding result appearing in [19] for the group $Sp(n) \cdot Sp(1)$. There, we first extended to the Hermitian quaternionic vector space V some notions and results of a vector space endowed with a complex structure (see [11]). In [3], considering a 2-plane $U \subset V$ spanned by the pair (L, M), it has been introduced

$$\Delta(U) = \mathcal{N}(\mathcal{IM}(U)) = \frac{\mathcal{N}[Im(L \cdot M)]}{mis^2(L \wedge M)} = \tag{15}$$
$$= \frac{< L, IM >^2 + < L, JM >^2 + < L, KM >^2}{mis^2(L \wedge M)}.$$

In particular, in case the basis L, M is orthonormal, $\Delta(U) = \mathcal{N}(L \cdot M)$. It is a real number belonging to the close interval $[0, 1]$ and equals 1 iff $\dim U^{\mathbb{H}} = 1$. It has been proved that the quantity $\Delta(U)$ is an intrinsic property of a 2-plane (see (10)) preserved by the action of the group $Sp(n) \cdot Sp(1)$ on V.

We called the angle $\delta(U) \in [0, \pi/2]$ such that $\cos^2\delta(U) = \Delta(U)$ the **characteristic deviation of the real 2-plane** $U \subset V$. Moreover

$$\Delta(U) = cos^2\delta(U) = \cos(\widehat{U, IU}) + \cos(\widehat{U, JU}) + \cos(\widehat{U, KU})$$

where $\cos(\widehat{U,IU})$ (resp. $\cos(\widehat{U,JU})$, $\cos(\widehat{U,KU})$) denotes the cosine of the angle between the pairs of 2-planes (U, IU) (resp. (U, JU), (U, KU)).

In Proposition (2.11) we showed that such quantity does not depend on the admissible basis (I, J, K) of $\mathcal{Q}$.

Generalizing the definition of the characteristic deviation given for a 2-plane, in ([19]) we find the following

Definition 3.6. *Let $(X_1, \ldots, X_m)$ be an orthonormal basis of an m-dimensional subspace U. Denote by $U_{rs} = L(X_r, X_s)$, $r \neq s = 1, \ldots, m$. We call the quantity*

$$\Delta(U) = \begin{pmatrix} m \\ 2 \end{pmatrix}^{-1} \sum_{r<s} \Delta(U_{rs}) \tag{16}$$

the ***characteristic deviation of the subspace*** *U^m.*

There we proved that the characteristic deviation is an intrinsic property of a subspace $U \subset V$ i.e. depends neither on the admissible basis of $\mathcal{Q}$ (it is a consequence of Proposition (2.11)) nor on the chosen orthonormal basis of U which determines the 2-planes U_{rs} in (16). We proved the following

Theorem 3.7. *[19] The characteristic deviation δ determines completely the orbit of the 2-plane $U \subset \mathbb{H}^n$ in the real Grassmannian $G_{\mathbb{R}}(2, 4n)$ under the action of $Sp(n) \cdot Sp(1)$*

Since for of a 2-dimensional complex subspace U one has $\Delta(U) = 1$ we can state the

Corollary 3.8. *The group $Sp(n) \cdot Sp(1)$ acts transitively on the set of 2-dimensional complex subspaces.*

3.3 Some results for isoclinic 4-dimensional subspaces

Let $I \in S(\mathcal{Q})$. We now consider the case of a pure I-complex subspace (U, I) of the Hermitian quaternionic vector space $(V^{4n}, \mathcal{Q}, <,>)$. In point (7) of the Claim (1.7) we observed that, if $K, K' \in I^{\perp} \cap S(\mathcal{Q})$ then $KU = K'U$. Furthermore the subspace KU is I-complex. We will deal first with a 4-dimensional pure complex subspace since, we will show later, it is the fundamental brick to determine the $Sp(n)$-orbit of The complex and Σ-complex subspaces.

As we will soon show, a 4-dimensional complex subspace U is isoclinic i.e. $U \in \mathcal{IC}^4$. The study of isoclinic subspaces is carried on in [21]. Here we briefly recall the results we need in this paper referring to [21] both for proofs and a deeper analysis.

Proposition 3.9. *[21] Let $A \in S(\mathcal{Q})$. The pair (U, AU) of 4-dimensional subspaces is isoclinic iff the matrix of ω^A w.r.t. the orthonormal basis (X_1, X_2, X_3, X_4) has the form*

$$\omega^A : \begin{pmatrix} 0 & a & b & c \\ -a & 0 & \pm c & \mp b \\ -b & \mp c & 0 & \pm a \\ -c & \pm b & \mp a & 0 \end{pmatrix}. \tag{17}$$

It is a matrix with orthogonal rows and columns whose square norms evidently equal the square cosine of the angle of isoclinicity θ^A between the pair (U, AU) i.e.

$$\cos\theta^A = \sqrt{a^2 + b^2 + c^2}.$$

Then we proved the following result which is valid only in dimension 4 (besides obviously in dimension 2 being all 2-planes isoclinic).

Proposition 3.10. *[21] Let U be a 4 dimensional subspace and (I, J, K) an admissible basis. Suppose the pairs (U, IU), (U, JU), (U, KU) are isoclinic and $\theta^I, \theta^J, \theta^K$ the respective angles of isoclinicity. Then for any $A = \alpha_1 I + \alpha_2 J + \alpha_3 K \in S(Q)$ the pair (U, AU) is isoclinic and therefore $U \in \mathcal{IC}^4$. The angle of isoclinicity θ^A between the pair (U, AU) is given by*

$$\cos^2\theta^A = -\frac{1}{4}Tr[(\alpha_1\omega^I + \alpha_2\omega^J + \alpha_3\omega^K)^2] = -\frac{1}{4}Tr[(\omega^A)^2] \tag{18}$$

We recall that a 4-dimensional complex subspace is never orthogonal unless it is totally complex (in which case it has a double orthogonality). Suppose that U is not an orthogonal subspace (w.r.t. (I, J, K)) and let

$$X_2 = \frac{I^{-1}Pr^{IU}X_1}{\cos\theta^I},\ Y_2 = \frac{J^{-1}Pr^{JU}X_1}{\cos\theta^J},\ Z_2 = \frac{K^{-1}Pr^{KU}X_1}{\cos\theta^K} \tag{19}$$

be the unitary vectors such that (X_1, X_2), (X_1, Y_2), (X_1, Z_2) are (orthonormal) standard bases of the standard 2-planes U^I, U^J, U^K of $\omega^I, \omega^J, \omega^K$ respectively they generate. The quantities $< X_1, IX_2 >, < X_1, JY_2 >, < X_1, KZ_2 >$ are the (non negative) cosines of the principal angles of the pairs $(U^I, IU^I), (U^J, JU^J), (U^K, KU^K)$ or equivalently the absolute value of the cosine of the I, J, K-Kähler angles of the 2-planes U^I, U^J, U^K respectively. Let (X_1, X_2, X_3, X_4) (resp. (X_1, Y_2, Y_3, Y_4), resp. (X_1, Z_2, Z_3, Z_4)) be a standard basis of the forms ω^I (resp. ω^J, resp. ω^K) with the common leading vector X_1. Let moreover denote by

$$\xi =< X_2, Y_2 >, \quad \chi =< X_2, Z_2 >, \quad \eta =< Y_2, Z_2 >$$

where $\xi, \chi, \eta \in [-1, 1]$.

Proposition 3.11. *[21] The cosines $\xi =< X_2, Y_2 >$, $\chi =< X_2, Z_2 >$, $\eta =< Y_2, Z_2 >$ are invariants of U.*

They are not an intrinsic properties of U i.e. they depend on the chosen admissible basis (see [21]).

Definition 3.12. *Let (I, J, K) be an admissible basis. The subspace $U \in \mathcal{IC}^4$ with angles $(\theta^I, \theta^I, \theta^K)$ is said to be a 2-planes decomposable subspace (or simply 2-planes decomposable) if it admits an orthogonal decomposition into a pair of 2-planes both isoclinic with their I, J, K-images with angles $(\theta^I, \theta^I, \theta^K)$ respectively.*

In a 2-plane decomposable subspace the values of $\xi, \chi, \eta = \xi\cdot\chi$ are all clearly equal to ± 1. In case of a 4 dimensional complex subspace U this happens iff U is totally complex. It is straightforward to verify that if $U \in \mathcal{IC}^4$ is 2-plane decomposable w.r.t (I, J, K) then it is 2-planes decomposable w.r.t. any admissible basis.

By using the invariants (ξ, χ, η), in [21] we determined the following equivalent expression for the angle of isoclinicity

$$\begin{aligned}\cos^2\theta^A &= \alpha_1^2\cos^2\theta^I + \alpha_2^2\cos^2\theta^J + \alpha_3^2\cos^2\theta^K + 2\xi\alpha_1\alpha_2\cos\theta^I\cos\theta^J \\ &\quad + 2\chi\alpha_1\alpha_3\cos\theta^I\cos\theta^K + 2\eta\alpha_2\alpha_3\cos\theta^J\cos\theta^K.\end{aligned} \tag{20}$$

Proposition 3.13. *[21] Let (I, J, K) be an admissible hypercomplex basis and $U \in \mathcal{IC}^4$ with angles of isoclinicity equal respectively to $(\theta^I, \theta^J, \theta^K)$. Then $S = \cos^2\theta^I + \cos^2\theta^J + \cos^2\theta^K$ is an intrinsic property of U not depending on the admissible basis.*

The previous property is more general in the sense that it is valid for all subspaces $U \in \mathcal{IC}$ regardless their dimension.

Dealing with a complex subspace U we have two possibilities: either U is not totally complex in which case none among ξ, χ, η equals ± 1 or U has a double orthogonality, is 2-planes decomposable and in particular $\xi = \chi = \eta = 1$.

In order to determine the canonical bases in this two cases, let consider first the case that none among ξ, χ, η equals ± 1. Let $X_1 \in U$ unitary and (X_2, Y_2, Z_2) as in (19). The subspaces $L(X_1, X_2), L(X_1, Y_2), L(X_1, Z_2)$ are respectively standard subspaces of $\omega^I, \omega^J, \omega^K$ restricted to U. Consider $L(X_2, Y_2)$ and the vectors X_4, Y_4 of such 2-plane such that (X_2, X_4) and (Y_2, Y_4) are a pair of orthonormal basis consistently oriented with (X_2, Y_2) (then $< X_2, Y_4 >< 0$).

Let then $X_3 = -\frac{I^{-1}Pr^{IU}X_4}{\cos\theta^I}$ be the unique vector such that $< X_3, IX_4 >= \cos\theta^I$ i.e. $L(X_3, X_4)$ is an ω^I-standard 2-plane and analogously $Y_3 = -\frac{J^{-1}Pr^{JU}Y_4}{\cos\theta^J}$ the unique vector such that $< Y_3, JY_4 >= \cos\theta^J$. Clearly the vectors X_3 and Y_3 belong to U.

Analogously we consider $L(X_2, Z_2)$ and the vectors $\tilde{X}_4, Z_4$ of such 2-plane such that $(X_2, \tilde{X}_4)$ and (Z_2, Z_4) are a pair of orthonormal basis consistently oriented with the pair (X_2, Z_2) (then $< X_2, Z_4 >< 0$). Again, let $\tilde{X}_3 = -\frac{I^{-1}Pr^{IU}\tilde{X}_4}{\cos\theta^I}$ be the unique vector such that $< \tilde{X}_3, I\tilde{X}_4 >= \cos\theta^I$ i.e. $L(\tilde{X}_3, \tilde{X}_4)$ is an ω^I standard 2-plane and $Z_3 = -\frac{K^{-1}Pr^{KU}Z_4}{\cos\theta^K}$ the unique vector such that $< Z_3, KZ_4 >= \cos\theta^K$ i.e. $L(Z_3, Z_4)$ is an ω^K standard 2-plane. The vectors $\tilde{X}_3$ and Z_3 belong to U. Proceeding in the same way considering the oriented 2-plane $L(Y_2, Z_2)$ we determine the pair $(\tilde{Y}_4, \tilde{Z}_4)$ and consequently $(\tilde{Y}_3, \tilde{Z}_3)$. With the above choices in [21] we proved the following

Proposition 3.14. *[21]*

$$X_3 = \frac{IPr^{IU}X_4}{\cos\theta^I} = \frac{JPr^{JU}Y_4}{\cos\theta^J} = Y_3, \qquad \tilde{X}_3 = \frac{IPr^{IU}\tilde{X}_4}{\cos\theta^I} = \frac{KPr^{KU}Z_4}{\cos\theta^K} = Z_3,$$
$$\tilde{Y}_3 = \frac{JPr^{IU}\tilde{Y}_4}{\cos\theta^J} = \frac{KPr^{KU}\tilde{Z}_4}{\cos\theta^K} = \tilde{Z}_3.$$

Definition 3.15. *Let $U \in \mathcal{IC}^4$, (I, J, K) be an admissible basis and $(\theta^I, \theta^J, \theta^K)$ the respective angles of isoclinicity. In case none among ξ, χ, η is equal to ± 1 (in particular if U is neither orthogonal nor 2-planes decomposable), for any unitary $X_1 \in U$, that we call leading vector, we define the following standard bases of $\omega^I|_U$*

and $\omega^J|_U$ *respectively*

$$\{X_i\} = \{X_1, X_2 = \frac{I^{-1}Pr^{IU}X_1}{\cos\theta^I}, X_3 = -\frac{I^{-1}Pr^{IU}X_4}{\cos\theta^I}, X_4 = \frac{Y_2-\xi X_2}{\sqrt{1-\xi^2}}\},$$
$$\{Y_i\} = \{X_1, Y_2 = \frac{(J^{-1}Pr^{JU}X_1)}{\cos\theta^J}, Y_3 = X_3 = -\frac{I^{-1}Pr^{JU}Y_4}{\cos\theta^J}, Y_4 = \frac{-X_2+\xi Y_2}{\sqrt{1-\xi^2}}\}$$

the ω^I *and* ω^J*-**chains** of* U *centered on* X_1*, and the following standard bases of* $\omega^I|_U$ *and* $\omega^K|_U$ *respectively*

$$\{\tilde{X}_i\} = \{X_1, X_2 = \frac{I^{-1}Pr^{IU}X_1}{\cos\theta^I}, \tilde{X}_3 = -\frac{I^{-1}Pr^{IU}\tilde{X}_4}{\cos\theta^I}, \tilde{X}_4 = \frac{Z_2-\chi X_2}{\sqrt{1-\chi^2}}\},$$
$$\{Z_i\} = \{X_1, Z_2 = \frac{K-1Pr^{KU}X_1}{\cos\theta^K}, Z_3 = \tilde{X}_3 = -\frac{K^{-1}Pr^{KU}Z_4}{\cos\theta^K}, Z_4 = \frac{-X_2+\chi Z_2}{\sqrt{1-\chi^2}}\}$$

the ω^I *and* ω^K*-**chains** of* U *centered on* X_1 *and the following standard bases of* $\omega^J|_U$ *and* $\omega^K|_U$ *respectively.*

$$\{\tilde{Y}_i\} = \{X_1, Y_2 = \frac{J^{-1}Pr^{JU}X_1}{\cos\theta^J}, \tilde{Y}_3 = -\frac{J^{-1}Pr^{JU}\tilde{Y}_4}{\cos\theta^J}, \tilde{Y}_4 = \frac{Z_2-\eta Y_2}{\sqrt{1-\eta^2}}\},$$
$$\{\tilde{Z}_i\} = \{X_1, Z_2 = \frac{K^{-1}Pr^{KU}X_1}{\cos\theta^K}, \tilde{Z}_3 = \tilde{Y}_3 = -\frac{K^{-1}Pr^{KU}\tilde{Z}_4}{\cos\theta^K}, \tilde{Z}_4 = \frac{-Y_2+\eta Z_2}{\sqrt{1-\eta^2}}\}$$

the ω^J *and* ω^K*-**chains** of* U *centered on* X_1*. We denote by* $\Sigma(X_1)$ *the set of the six chains with leading vector* X_1*.*

Clearly $\Sigma(X_1)$ is uniquely determined by the leading vector X_1.

In case a pair among (ξ, χ, η) and hence all three pairs are equal to ± 1, (namely either $\eta = \xi = \chi = 1$ or two of them are equal to -1 and the other to 1), we give the following

Definition 3.16. *In case* U *is a 2-planes decomposable subspace i.e.* ξ, χ, η *are all equal to* ± 1 *we define the following chains:*

$$\{X_i\} = \{\tilde{X}_i\},\ \{Y_i\} = (X_1, \xi X_2, X_3, \xi X_4) = \{\tilde{Y}_i\},$$

$$\{Z_i\} = (X_1, \chi X_2, X_3, \chi X_4) = (X_1, \eta Y_2, X_3, \eta Y_4) = \{\tilde{Z}_i\}.$$

In particular if U *has a double orthogonality (which happens in particular if* U *is totally complex) or a triple orthogonality (iff* U *is a totally real subspace) one has*

$$\{X_i\} = \{\tilde{X}_i\} = \{Y_i\} = \{\tilde{Y}_i\} = \{Z_i\} = \{\tilde{Z}_i\}.$$

Clearly $L(X_3, X_4) = L(\tilde{X}_3, \tilde{X}_4)$. The bases (X_3, X_4) and $(\tilde{X}_3, \tilde{X}_4)$, being ω^I-standard bases, are consistently oriented. Let

$$C : \begin{pmatrix} <X_3, \tilde{X}_3> & <X_3, \tilde{X}_4> \\ <X_4, \tilde{X}_3> & <X_4, \tilde{X}_4> \end{pmatrix} = \begin{pmatrix} \Gamma & -\Delta \\ \Delta & \Gamma \end{pmatrix}$$

the orthogonal matrix of the change of basis. The orthogonal matrices $C_{IJ} = (<X_i, Y_j>)$ and $C_{IK} = (<X_i, Z_j>)$ of the relative position of the basis $\{X_i\} =$

(X_1, X_2, X_3, X_4), $\{Y_i\} = (X_1, Y_2, X_3, Y_4)$ and $\{Z_i\} = (X_1, Z_2, \tilde{X}_3, Z_4)$ are given by

$$
\begin{aligned}
C_{IJ} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \xi & 0 & -\sqrt{1-\xi^2} \\ 0 & 0 & 1 & 0 \\ 0 & \sqrt{1-\xi^2} & 0 & \xi \end{pmatrix}, \\
C_{IK} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \chi & 0 & -\sqrt{1-\chi^2} \\ 0 & -\Delta\sqrt{1-\chi^2} & \Gamma & -\chi\Delta \\ 0 & \Gamma\sqrt{1-\chi^2} & \Delta & \chi\Gamma. \end{pmatrix}.
\end{aligned}
\tag{21}
$$

To determine $\Gamma =< X_3, \tilde{X}_3 >$, being $\overline{< Y_2, Z_2 >} =< Y_2, X_2 >< Z_2, X_2 > + < Y_2, X_4 >< Z_2, X_4 >$, we get $\eta = \xi\chi + \sqrt{1-\xi^2}\sqrt{1-\chi^2}\,\Gamma$. From the above expression, in case neither ξ nor χ equal 1, we get:

$$
\Gamma = \frac{\eta - \xi\chi}{\sqrt{1-\xi^2}\,\sqrt{1-\chi^2}} \in [-1, 1]. \tag{22}
$$

Proposition 3.17. *[21] If none among ξ, χ, η is equal to ± 1 the value of $\Gamma \in [-1, 1]$ is given in (22). If instead at least one among ξ, χ, η is equal to ± 1 then $\Gamma = 1$. In particular this happens if U is orthogonal or is a 2-planes decomposable subspace. In all cases, the pair (Γ, Δ) is an invariant of U.*

Proposition 3.18. *[21] The matrices C_{IJ} and C_{IK} given in (21) w.r.t. the chains $\{X_i\}, \{Y_i\}$ and $\{X_i\}, \{Z_i\}$ centered on a common leading vector are invariant of U.*

Since if U is orthogonal one has $(\Gamma, \Delta) = (1, 0)$, we can deduce the following

Corollary 3.19. *In case of double or triple orthogonality one has $C_{IJ} = C_{IK} = Id$.*

Following the definition given in [20], we give the

Definition 3.20. *Let $U \in \mathcal{IC}^4$. Fixed an admissible basis (I, J, K), for any leading vector X_1, we call the chains $\{X_i\}, \{Y_i\}, \{Z_i\}$ (resp. the matrices C_{IJ} and C_{IK}) determined above* ***canonical bases*** *(resp.* ***canonical matrices****) of the subspaces $U \in \mathcal{IC}^4$.*

Clearly for any leading vector we have a different set of canonical bases. As explained beforehand, we denote them "canonical" since, by the invariance of $(\xi, \chi, \eta, \Delta)$, the matrices C_{IJ} and C_{IK} are invariants of $U \in \mathcal{IC}^4$ having the unique forms given in (21) regardless the leading vector X_1. We summarize the results obtained in the following

Proposition 3.21. *[21] Fixed an admissible basis (I, J, K), to any $U \in \mathcal{IC}^4$ we can associate the orthogonal canonical matrices C_{IJ} and C_{IK} given in (21) representing the mutual position of the canonical (standard) bases $\{X_i\}, \{Y_i\}, \{Z_i\}$ of $\omega^I, \omega^J, \omega^K$. Such matrices depend on the triple of invariants (ξ, χ, η) and on the sign of $\Delta =< X_4, \tilde{X}_3 >= \pm\sqrt{1-\Gamma^2}$ where $\Gamma = \Gamma(\xi, \chi, \eta)$ is given in (22) if none among ξ, χ, η is equal to ± 1 else $\Gamma = 1$. The second case happens in particular if U is orthogonal or a 2-planes decomposable subspace.*

Then according to Theorem (3.2) we state the

Theorem 3.22. *[21] The invariants $(\xi, \chi, \eta, \Delta)$ together with the angles $(\theta^I, \theta^J, \theta^K)$ determine the orbit of any $U \in \mathcal{IC}^4$. In particular if U is orthogonal or 2-planes decomposable (in which case $(\Gamma, \Delta) = (1, 0)$) the first set reduces to the pair (ξ, χ).*

A logical consequence of the above theorem is the following

Corollary 3.23. *If the pair of subspaces (U, W) share the same invariants $(\xi, \chi, \eta, \Gamma, \Delta)$ and the same angles of isoclinicity $(\theta^I, \theta^J, \theta^K)$ w.r.t. the admissible basis (I, J, K) then they share the same invariants and angles w.r.t. any admissible basis.*

3.4 The 4-dimensional complex subspace

Proposition 3.24. *Let (U, I) be a 4-dimensional complex subspace. Then $U \in \mathcal{IC}^4$. Moreover for any pair $X, Y \in U$ belonging to orthogonal I-complex 2-planes (i.e. $L(X, IX) \perp L(Y, IY)$), the characteristic angle (i.e. the angle $\widehat{QX, QY}$) is an invariant of U and equals the Euclidean angle of the pair (U, KU).*

Proof. For any unitary pair $X, Y \in U$ with $Y \in L(X, IX)^\perp$, the set $\{X, IX, Y, IY\}$ is an orthonormal basis of U. Let (I, J, K) be an adapted basis. Being $U = IU$, the pair (U, IU) is isoclinic with cosine of the angle of isoclinicity equal to 1. Moreover, from point (7)of the Proposition (1.7), one has $JU = KU$. We compute the principal angles $\theta_i \in [0, \pi/2],\ i = 1, \ldots, 4$ between the I-complex 4-planes U and KU recalling that their square cosines are the eigenvalues of the symmetric matrix GG^t where by G we denote the Gram matrix of $U \times KU$ (matrix of the orthogonal projector $Pr^U : KU \to U$). Such matrix, w.r.t. the orthonormal basis (X, IX, Y, IY) of U and (KX, JX, KY, JY) of KU, assumes the form

$$G = \begin{array}{c} \begin{array}{cccc} \quad KX \quad & \quad JX \quad & \quad KY \quad & \quad JY \quad \end{array} \\ \left(\begin{array}{cccc} 0 & 0 & <X, KY> & <X, JY> \\ 0 & 0 & <X, JY> & -<X, KY> \\ -<X, KY> & -<X, JY> & 0 & 0 \\ -<X, JY> & <X, KY> & 0 & 0 \end{array}\right) \end{array}$$

therefore, according to Proposition (3.9), the pair of 4-dimensional I-complex subspaces (U, KU) is isoclinic and according to the Proposition (3.10) one has that

$U \in \mathcal{IC}^4$. Denoting by $a =< X, KY >$ and $b =< X, JY >$, one has

$$G^t G = \begin{pmatrix} a^2+b^2 & 0 & 0 & 0 \\ 0 & a^2+b^2 & 0 & 0 \\ 0 & 0 & a^2+b^2 & 0 \\ 0 & 0 & 0 & a^2+b^2 \end{pmatrix}.$$

So, according to the Definition (2.7), the angle of isoclinicity θ^K of the complex 4-planes U and KU or equivalently one of the four identical principal angles is given by

$$\cos \theta^K = \sqrt{< X, KY >^2 + < X, JY >^2}.$$

From Proposition (2.6) one has

$$\cos \theta_1 \cdot \cos \theta_2 \cdot \cos \theta_3 \cdot \cos \theta_4 = \cos (\widehat{U, KU}) = (< X, KY >^2 + < X, JY >^2)^2 \quad (23)$$

which clearly depends only on U and KU. We deduce that $(< X, JY >^2 + < X, KY >^2)$ is an invariant of U for any I-orthonormal pair $X, Y \in U$ (i.e. for any $Y \in L(X, IX)^\perp \cap U$). This implies that, for any I-orthonormal pair $X, Y \in U$, the characteristic angle $\theta = \widehat{\mathcal{Q}X, \mathcal{Q}Y}$ is an invariant of U as well. In fact, if $X' = AX$, $Y' = AY$ where A is an orthogonal map commuting with I in order to preserve the I-orthogonality between X', Y',

$$\begin{aligned} \cos \widehat{\mathcal{Q}X, \mathcal{Q}Y} &= (< X, Y >^2 + < X, IY >^2 + < X, JY >^2 + < X, KY >^2)^2 \\ &= (< X, JY >^2 + < X, KY >^2)^2 \\ &= \cos (\widehat{U, KU}) = (< X', JY' >^2 + < X', KY' >^2)^2 = \cos \widehat{\mathcal{Q}X', \mathcal{Q}Y'}. \end{aligned} \quad (24)$$

□

Definition 3.25. *Let (U, I) be a 4-dimensional I-complex subspace. We call* ***$I^\perp$-Kähler angle*** *of a 4-dimensional I-complex subspace U, and denote it by $\theta^{I^\perp}(U)$, the angle of isoclinicity of the pair (U, KU), for any $K \in I^\perp$. One has*

$$\cos \theta^{I^\perp}(U) = \sqrt{< X, KY >^2 + < X, JY >^2} = \cos \theta^J = \cos \theta^K$$

where (I, J, K) is any adapted basis and (X, Y) unitary with $Y \in L(X, IX)^\perp \cap U$. We denote such subspace by the triple $(U, I, \theta^{I^\perp})$.

Let consider the case that (U, I) is not totally complex i.e. $\theta^K \neq \pi/2$. it is easily seen that given the unitary vectors X, Y of U with $Y \in L(X, IX)^\perp \cap U$, and denoted by $a =< X, KY >$ and $b =< X, JY >$, the unitary vector $Z_2 \in U$ such that X and KZ_2 are associated left and right singular vectors of the pair (U, KU) i.e. $Pr^{KU} X = \cos \theta^K K Z_2$ and $Pr^U (KZ_2) = \cos \theta^K X$ is given by

$$Z_2 = \frac{a}{\cos \theta^K} Y + \frac{b}{\cos \theta^K} IY. \quad (25)$$

Clearly it does not depend on the orthonormal pair (Y, IY); it belongs to the ω^I-standard I-complex 2-plane $L(X, IX)^\perp = L(Y, IY)$. We can then state the following

Corollary 3.26. *Consider $(U, I, \theta^{I^\perp} \neq \pi/2)$. For any complex structure $K \in \mathcal{Q}$ anticommuting with I, the orthogonal projection $Pr^{KU}X \in KU$ of a unitary vector $X \in U$ is such that the unitary vectors $(X, Z_2 = \frac{K^{-1}Pr^{KU}X}{\cos\theta^K})$ belong to strictly orthogonal I-complex 2-planes.*

Clearly (X, KZ_2) are related principal vectors of the pair of 4-dimensional I-complex subspaces (U, KU). Considering a different complex structure $\bar{K} \in I^\perp$, the vector $Z_2 = \frac{\bar{K}^{-1}Pr^{\bar{K}U}X}{\cos\theta^K}$ changes inside the I-complex 2-plane $L(X, IX)^\perp \cap U$. In particular one has that, for any adapted basis (I, J, K), the vector $Y_2 = \frac{J^{-1}Pr^{KU}(X)}{\cos\theta^J} = -IZ_2$ and $Z_2 = \frac{K^{-1}Pr^{KU}X}{\cos\theta^K}$ form an orthonormal basis of $L(X, IX)^\perp$.

From the isoclinicity of U and KU any orthonormal basis in U is a basis of singular vectors as well as any orthonormal basis in KU is a basis of right singular vectors. Clearly to any such basis in U corresponds only one basis of right singular vector in KU in order to have a pair of *related bases* ([10]).

With respect to these related bases, the Gram matrix $G(U \times KU)$ is diagonal with non negative diagonal entries equal to $\cos\theta^K = \cos\theta^J = \cos\theta^{I^\perp}$. The existence of such diagonal form is stated in [1] by Theorem (2.4) and follows from the theory of the singular values decomposition applied to the matrix G.

In case instead (U, I) is totally complex, for any $X \in U$, one has that $Pr^{KU}X = \mathbf{0}$. Being the ω^I standard 2-plane with leading vector X_1 given by $L(X_1, X_2 = -IX_1)$, in this case we assume that $Z_2 = X_2$ for any $K \in I^\perp$

Observe that the cosine of the characteristic angle $\cos\widehat{\mathcal{Q}X, \mathcal{Q}Y}$ equals the square cosine of the angle between the I-complex planes $\mathbb{C}X = L(X, IX)$ and $\mathbb{C}(KY) = L(KY, JY)$ i.e. for any $X, Y \in U$ such that $\mathbb{C}X \perp \mathbb{C}Y$ it is

$$\cos\widehat{\mathcal{Q}X, \mathcal{Q}Y} = \cos(\widehat{U, KU}) = (< X, KY >^2 + < X, JY >^2)^2 = \cos^2(\widehat{\mathbb{C}X, \mathbb{C}(KY)})$$

for any adapted basis (I, J, K). In particular if (U, I) is totally complex one has that $\cos\widehat{\mathcal{Q}X, \mathcal{Q}Y} = \cos(\widehat{U, KU}) = 0$.

Recalling the definition of S introduced in Proposition (3.13), one has the following

Proposition 3.27. *There exists a 1:1 correspondence between the characteristic deviation $\Delta(U)$ of a 4-dimensional I-complex subspace $(U, I, \theta^{I^\perp})$ and the $I^\perp$-Kähler angle $\theta^{I^\perp} \in [0, \pi/2]$. It is given by*

$$\Delta(U, I, \theta^{I^\perp}) = \frac{2\cos^2\theta + 1}{3} = \frac{\cos^2(U, IU) + \cos^2(U, JU) + \cos^2(U, KU)}{3} = \frac{S}{3} \quad (26)$$

Proof. It follows from (16). Consider the orthonormal basis (X, Z_2, IX, IZ_2) where the pair (X, KZ_2) are related left and right singular vectors of (U, KU). One has

$$\begin{aligned}\Delta(U, I, \theta^{I^\perp}) &= \tfrac{1}{6}(\Delta(X, Z_2)_{\mathbb{R}} + \Delta(X, IX)_{\mathbb{R}} + \Delta(X, IZ_2)_{\mathbb{R}} \\ &\quad +\Delta(Z_2, IX)_{\mathbb{R}} + \Delta(Z_2, IZ_2)_{\mathbb{R}} + \Delta(IX, IZ_2)_{\mathbb{R}}) = \\ &= \tfrac{1}{6}(< X, KZ_2 >^2 +1+ < X, KZ_2 >^2 + < X, KZ_2 >^2 +1 \\ &\quad + < X, KZ_2 >^2) = \tfrac{2\cos^2\theta^{I^\perp}+1}{3} = \tfrac{S}{3}\end{aligned}$$

□

3.5 The associated plane of a 4-dimensional complex subspace

The standard form of the skew-symmetric form $\omega^K : (X \times Y) \to < X, KY >$ restricted to the 4-dimensional pure subspace $(U, I, \theta^{I^\perp})$, for any $K \in I^\perp$, determines a decomposition $U = U_1 \overset{\perp}{\oplus} U_2$ into an orthogonal sum of a pair of K-orthogonal ω^K-standard 2-planes (i.e. $U_1 \perp KU_2$). If $U_1 = L(X_1, X_2)$ and $U_2 = L(X_3, X_4)$ where the bases are orthonormal, clearly (X_1, X_2, X_3, X_4) and $(KX_2, -KX_1, KX_4, -KX_3)$ are bases of related left and right singular vectors of the projector $Pr^{KU} : U \to KU$ or equivalently of related principal vectors of the pair of subspaces (U, KU).

Let now consider the case that the 4-dimensional pure subspace (U, I) is not totally complex. In this case, from the Corollary (3.26) it is clearly $U_2 = IU_1$ and consequently $U_1 \perp IU_1$. Then, if J completes the adapted basis (I, J, K), it is also $U_1 \perp JU_1$. Therefore, for any $K' \in I^\perp$, given an $\omega^{K'}$-standard 2-plane $U_1 \subset U$ one has that $U = U_1 \overset{\perp}{\oplus} IU_1 2$ is an $\omega^{K'}$-standard decomposition of U into orthogonal standard 2-planes.

Definition 3.28. *We call **associated plane** to the I-complex 4-dimensional pure subspace $(U, I, \theta^{I^\perp} \neq \pi/2)$ any 2-dimensional plane $U' \subset U$ characterized by the existence of an orthonormal basis, say (X, Z), such that the vectors (X, KZ) are a pair of related principal vectors of the pair (U, KU) for some $K \in I^\perp$.*

If any such basis exists, all consistently oriented orthonormal bases have the same property. Clearly $< X, KZ >= \cos\theta^{I^\perp}$. Being U pure, one has that U' is necessarily totally real. From the Corollary (3.26), $Z(K) = \frac{K^{-1}Pr^{KU}X}{\cos\theta^K} \in L(X, IX)^\perp$.
For any $X \in U$ and $K \in I^\perp$, all subspaces $U'(K) = L(X, Z(K))$ where $Z(K) = \frac{K^{-1}Pr^{KU}X}{\cos\theta^K}$ are associated plane of $(U, I, \theta^{I^\perp} \neq \pi/2)$. When we need to specify the structure $K \in S(\mathcal{Q})$, we denote the associated plane U' as $U'(K)$.

Proposition 3.29. *Let $(U, I, \theta^{I^\perp})$ be a 4-dimensional I-complex subspace with $I^\perp$-Kähler angle $\theta^{I^\perp} \neq \pi/2$. The 2-plane $U' = L(X, Z) \subset U$ is an associated plane iff any of the following equivalent conditions are satisfied.*

1. *There exists $K \in I^\perp$ such that $Pr_U^{KU} U' = KU'$.*

   Proof. If $U'(K)$ is an associated plane and (X, Z) an orthonormal basis such that the pair (X, KZ) are related principal vectors of the pair (U, KU), then

$Pr^{KU}X = \cos\theta^{I^\perp} KZ$ and by the skew-symmetry of ω^K one has $Pr^{KU}Z = \cos\theta^{I^\perp}(-KX)$. Then $Pr^{KU}U' = KU'$. Viceversa, if there exists some $K \in I^\perp$ such that $Pr^{KU}U' = KU'$, let (X, Y) be an orthonormal basis of U'. One has that $Pr^{KU}X =< X, KY > KY$ with clearly $< X, KY >= \pm\cos\theta^{I^\perp}$. According to the sign, either (X, KY) or (Y, KX) are a pair of related principal vectors of the pair (U, KU). $\square$

2. $(U, I, \theta^{I^\perp}) = U' \overset{\perp}{\oplus} IU'$.

Proof. Let $U' = U'(K) = L(X, Z)$ be an associated plane of U. Clearly also $IU' = IU'(K)$ is an associated plane. Furthermore, by the Corollary (3.26), the vectors (X, Z, IX, IZ) form an orthonormal basis which implies that the direct sum $U = U' \oplus IU'$ is orthogonal. Viceversa, let $(U, I, \theta^{I^\perp}) = U' \overset{\perp}{\oplus} IU'$. By the Proposition (1.7) point (4), the 2-plane $U' = L(X, Z)$ is totally real. W.r.t. the orthonormal basis (X, Z, IX, IZ) one has $Pr^{KU}X =< X, JZ > Z+ < X, KZ > IZ$ then $< X, K'Z >= \cos\theta^{I^\perp}$ for $K' = \frac{1}{\cos\theta^{I^\perp}} < X, JZ > J+ < X, KZ > K$. $\square$

3. *There exists some $K \in I^\perp$ w.r.t. which U' is a standard 2-plane of $\omega^K|_U$.*

Proof. If $U'(K)$ is an associated plane then, w.r.t. to the decomposition $U = U' \overset{\perp}{\oplus} IU'$, $\omega^K|_U$ assumes standard form. Viceversa in the decomposition $U = U_1 \oplus IU_1$ associated to the form $\omega^K|_U$, the standard 2-plane U_1 is an associated plane of $(U, I, \theta^{I^\perp})$ since $Pr^{KU}U_1 = KU_1$ and the conclusion follows from point (1). $\square$

4. *There exists $K \in I^\perp$ such that the K-Kähler angle of U' equals the $I^\perp$-Kähler angle $\theta^{I^\perp}$.*

Proof. By the isoclinicity of the pair (U, KU), the angle between any vector $X \in U$ and the subspace KU equals the $I^\perp$-Kähler angle $\theta^{I^\perp}$. if $U'(K)$ is an associated plane, from 1) $Pr^{KU}U' = KU'$, then both singular values of the projector Pr^{KU} restricted to U' equal $\cos\theta^{I^\perp}$. Viceversa if there exists $K \in I^\perp$ such that the K-Kähler angle of U' equals $\theta^{I^\perp}$ clearly $Pr^{KU}U' = KU'$ i.e., from previous point (1), U' is an associated plane. $\square$

5. *$\mathcal{IM}(U') = aj + bk$ where $j, k \in i^\perp$ and orthonormal with $a^2 + b^2 = \cos^2\theta^{I^\perp}$ i.e. iff $\mathcal{IM}(U') \in i^\perp$, $\Delta(U') = a^2 + b^2 = \cos^2\theta^{I^\perp}$.*

Proof. If $U'(K) = L(X, Z)$ is an associated plane of $(U, I, \theta^{I^\perp})$ with $< X, KZ >= \cos\theta^{I^\perp}$, then $U' \perp IU'$, $U' \perp JU'$ where J completes the adapted basis (I, J, K).After identifying the adapted hypercomplex structures (I, J, K) of V

with (R_{-i}, R_{-j}, R_{-k}) of $\mathbb{H}$, one has $\mathcal{IM}(U') =< X, IZ > i+ < X, JZ > j+ < X, KZ > k = \cos\theta k$. Using a different adapted basis the results follows.

Viceversa if $U' \subset (U, I, \theta^{I^\perp})$, with $\mathcal{IM}(U') = aj + bk$ w.r.t. the adapted basis (I, J, K), one has that U' is an associated plane w.r.t. $K' = aJ + bK$, $a^2 + b^2 = 1$. In fact $< X, K'Z >= \cos\theta^{I^\perp}$. In case $< X, KZ >< 0$ we consider the orthonormal basis $(X, -Z)$ or any other with the same orientation. From 3), $U'(K')$ is an associated plane. □

6. *One and hence any orthonormal basis of U', say (X, Z) is I-orthogonal i.e. $Z \in L(X, IX)^\perp$.*

   Proof. If $U'(K) = L(X, Z)$ is an associated plane then for any $X \in U'$ from the Corollary (3.26), one has $Z = \frac{K^{-1}Pr^{KU}X}{\cos\theta^K} \in L(X, -IX)^\perp$. Then $Pr^{KU}(U') = KU'$ and the conclusion follows from point (1). Viceversa if the unitary basis (X, Z) is I-orthogonal, then (X, Z, IX, IZ) is an orthonormal basis of U and (KX, KZ, JX, JZ) of KU w.r.t. which $Pr^{KU}X =< X, KZ > KZ+ < X, JZ > JZ$ with $< X, KZ >^2 + < X, JZ >^2 = \cos^2\theta^{I^\perp}$. Then $U' = U'(K')$ with $K' = \frac{1}{\cos\theta^{I^\perp}} < X, KZ > K+ < X, JZ > J$ is an associated plane since $< X, K'Z >= \cos\theta^{I^\perp}$. □

We summarize the above characterizations of the associated subspace in the following

Proposition 3.30. *Let $(U, I, \theta^{I^\perp} \neq \pi/2)$ be a 4-dimensional pure I-complex subspace. Then, for any $X \in U$, and $K \in I^\perp$ one has $U = U_1(K) \oplus IU_1(K)$ is direct orthogonal sum of the uniquely defined K-orthogonal associated planes $U_1 = L(X, Z)$ and $U_2 = IU_1 = L(IZ, IX)$ where $Z = \frac{K^{-1}Pr_U^{KU}(X)}{\cos\theta^{I^\perp}}$. The $I^\perp$-Kähler angle $\theta^{I^\perp}$ is the same as the K-Kähler angle $\Theta^K(U_1) = \Theta^K(U_2)$ of the associated planes U_1 and $U_2 = IU_1$ i.e.*

$$\cos\theta^{I^\perp} = \cos\Theta^K(U_1) =< X, KZ >=< IZ, KIX >= \cos\Theta^K(U_2).$$

Moreover

$$\mathcal{IM}(U_1(K)) = \mathcal{IM}(U_2(K)) = \cos\theta k = \cos\theta^{I^\perp}.$$

Observe that the strictly orthogonal associated planes $U = U_1(K)$ and $IU_1 = U_2(K)$ are only K-orthogonal but clearly never I orthogonal (furthermore they are not J orthogonal unless U is totally complex i.e. $\cos\theta^K = 0$). In particular they are never orthogonal in Hermitian sense. Namely, since any totally real 2-plane never belong to a quaternionic line but to a quaternionic 2-plane, we have that the pair of quaternionic planes containing U_1 and U_2 (eventually coinciding) are never orthogonal. We conclude this section with the

Proposition 3.31. *Given $(U, I, \theta^{I^\perp})$ and the associated plane $U_1(K')$, $K' \in I^\perp$ of $(U, I, \theta^{I^\perp})$, the $A^\perp$-Kähler angle $\theta^{A^\perp}$ of the A-complex subspace $\bar{U} = U_1(K') \oplus AU_1(K')$ with $A = aI + b(\alpha J + \beta K)$, $a^2 + b^2 = \alpha^2 + \beta^2 = 1$ equals $\theta^{I^\perp}$.*

Proof. It follows from the fact that for $A \in K'^{\perp}$, $U_1(K')$ is an associated plane of $\bar{U}$ as well. In fact, let (I, J, K) be an adapted basis and consider for instance $U_1(K)$. Being $U_1(K) \perp IU_1(K)$ and $U_1(K) \perp JU_1(K)$ one has that $U_1(K) \perp J'U_1(K)$, $\forall J' \in L(I, J)$. Consequently $Pr^{KU}U_1(K) = KU_1(K)$. Extending such results to all $K' \in L(J, K)$ the conclusion follows. □

Then, given $(U, I, \theta^{I^{\perp}})$, for any $A \in S(\mathcal{Q})$ we can build an A-complex 4-dimensional subspace with $\theta^{A^{\perp}} = \theta^{I^{\perp}}$.

As an example, let consider $I' = \frac{1}{3}I + \frac{2}{3}J + \frac{2}{3}K$. Then $I' \in L(I, J')$ with $J' = \frac{1}{\frac{2\sqrt{2}}{3}}(\frac{2}{3}J + \frac{2}{3}K)$, In this case, $a = \frac{1}{3}$, $b = \frac{2\sqrt{2}}{3}$, $\alpha = \beta = \frac{1}{\sqrt{2}}$. The complex structure in $L(J, K)$ orthogonal to J' is $K' = \frac{1}{\frac{2\sqrt{2}}{3}}(-\frac{2}{3}J + \frac{2}{3}K)$ and consequently $\bar{Z}_2 = \frac{K'^{-1}Pr^{KU}X_1}{\cos\theta^{I^{\perp}}} = \frac{1}{\frac{2\sqrt{2}}{3}}(\frac{2}{3}IZ + \frac{2}{3}Z) = \frac{1}{\sqrt{2}}(IZ + Z)$ and the associated plane is $U(K') = L(X_1, \bar{Z}_2)$. It is straightforward to verify that $\bar{U} = U_1(K') \oplus I'U_1(K')$ is I'-complex with $I'^{\perp} = I^{\perp}$ i.e. $\bar{U} = (\bar{U}, I', \theta^{I^{\perp}})$. In fact, considering the adapted basis $(I', \bar{J}, \bar{K})$ with $\bar{J} = \frac{2}{3}I + \frac{1}{3}J - \frac{2}{3}K$ and $\bar{K} = -\frac{2}{3}I + \frac{2}{3}J - \frac{1}{3}K$ one has

$$
\begin{aligned}
< X, \bar{K}\bar{Z}_2 > \quad &= \quad < X, (-\tfrac{2}{3}I + \tfrac{2}{3}J - \tfrac{1}{3}K)(\tfrac{1}{\sqrt{2}}(IZ + Z)) > \tfrac{1}{\cos\theta^{I^{\perp}}} = \\
&\quad -\tfrac{1}{\sqrt{2}} < X_1, KZ >= -\tfrac{1}{\sqrt{2}}\cos\theta^{I^{\perp}} \\
< X, \bar{J}\bar{Z}_2 > \quad &= \quad < X, (\tfrac{2}{3}I + \tfrac{1}{3}J + -\tfrac{2}{3}K)(\tfrac{1}{\sqrt{2}}(IZ + Z)) > \tfrac{1}{\cos\theta^{I^{\perp}}} = \\
&\quad -\tfrac{1}{\sqrt{2}} < X_1, KZ >= -\tfrac{1}{\sqrt{2}}\cos\theta^{I^{\perp}}
\end{aligned}
$$

which implies that $\cos\theta^{I'^{\perp}} = \sqrt{< X, \bar{J}\bar{Z}_2 >^2 + < X, \bar{K}\bar{Z}_2 >^2} = \cos\theta^{I^{\perp}}$.

What stated in Proposition (3.31) will be relevant when studying the $Sp(n) \cdot Sp(1)$-orbits in the real Grassmannian. In an article that we will publish soon we will show that the $I^{\perp}$-Kähler angle $\theta^{I^{\perp}}$ of an I-complex 4-dimensional subspace of a quaternionic Hermitian vector space constitutes the full system of invariant for its $Sp(n) \cdot Sp(1)$-orbit. Then, from Proposition (3.31), we have that, given a 2-plane U' with $\mathcal{IM}(U') = \cos\theta k$, which as stated in point (5) of the Proposition (3.29) is an associated plane of $U = U' \oplus IU'$, all subspaces $U_A = U' \oplus AU'$ with $A = aI + b(\alpha J + \beta K)$, $a^2 + b^2 = \alpha^2 + \beta^2 = 1$ are in the same $Sp(n) \cdot Sp(1)$-orbit.

3.6 Canonical bases and canonical matrices of a 4-dimensional complex subspace

Let $(U, I, \theta^{I^{\perp}})$ be a 4-dimensional I-complex subspace and (I, J, K) an adapted basis. Recall that $JU = KU$. Using the same notations that appear in [21], for any unitary $X_1 \in U$, we denote by $X_2 = I^{-1}Pr^{IU}X$, by $Y_2 = \frac{J^{-1}Pr^{JU}X}{\cos\theta^{I^{\perp}}}$ and by $Z_2 = \frac{K^{-1}Pr^{KU}X}{\cos\theta^{I^{\perp}}}$. Clearly $X_2 = -IX_1$. The pair $(X_1, -IX_1)$ is an ω^I-standard basis of $U_1 = L(X, -IX)$. Clearly $Pr^{IU}U_1 = IU_1 = U_1$. Furthermore we denote by $\xi =< X_2, Y_2 >$, $\chi =< X_2, Z_2 >$, $\eta =< Y_2, Z_2 >$. According to the Proposition (3.11) such triple is an invariant of U.

Proposition 3.32. *Let $(U, I, \theta^{I^\perp})$ be a 4-dimensional I-complex subspace not totally complex and (I, J, K) an adapted basis. Choose $X_1 \in U$ unitary and let $U_1 = L(X_1, -IX_1)$ be the ω^I-standard plane. The vectors $Y_2 = \frac{J^{-1}Pr^{JU}X}{\cos\theta^{I^\perp}} \in U_1^\perp$, $Z_2 = \frac{K^{-1}Pr^{KU}X}{\cos\theta^{I^\perp}} \in U_1^\perp$. Moreover one has $\xi = \chi = \eta = 0$ and consequently*

$$(X_1, X_2 = I^{-1}Pr^{IU}X = -IX_1, \quad Y_2 = \frac{J^{-1}Pr^{JU}X}{\cos\theta^{I^\perp}}, \quad Z_2 = \frac{K^{-1}Pr^{KU}X}{\cos\theta^{I^\perp}})$$

is an orthonormal basis of U and, upon reordering, they form a triple of standard bases $\{X_i\}, \{Y_i\}, \{Z_i\}$ centered on X_1 of respectively $\omega^I|_U, \omega^J|_U, \omega^K|_U$. Furthermore such triple are exactly the $\{X_i\}, \{Y_i\}, \{Z_i\}$ chains centered on X_1 defined in (3.15). W.r.t. such chains the orthogonal matrices $C_{IJ} = < X_i, Y_j >$, $C_{IK} = < X_i, Z_j >$ are given in (28) and, as stated in the Proposition (3.18), are invariants of $(U, I, \theta^{I^\perp})$. Furthermore they do not depend on the adapted basis.

Proof. From Corollary (3.26) one has that the unitary vectors $Y_2 = \frac{J^{-1}Pr^{JU}X}{\cos\theta^J} \in U_1^\perp$ and $Z_2 = \frac{K^{-1}Pr^{KU}X}{\cos\theta^K} \in U_1^\perp$ which implies that $\xi =< X_2, Y_2 >= 0$ as well as $\chi =< X_2, Z_2 >= 0$. Moreover $Y_2 = \frac{J^{-1}Pr^{JU}X_1}{\cos\theta^J} = \frac{J^{-1}KX_1\cos\theta^K}{\cos\theta^J} = -IZ_2$ which implies that $\eta =< Y_2, Z_2 >= 0$. To obtain the chains $\{X_i\}, \{Y_i\}, \{\tilde{X}_i, \{Z_i\}\}$ it is straightforward to verify that

$$X_4 = Y_2,\ Y_4 = -X_2,\ \tilde{X}_4 = Z_2,\ Z_4 = -X_2.$$

Furthermore

$$\tilde{X}_3 = -I^{-1}Pr^{IU}\tilde{X}_4 = I\tilde{X}_4 = IZ_2.$$

$$Y_2 = \frac{J^{-1}Pr^{JU}X_1}{\cos\theta} = \frac{J^{-1}Pr^{KU}X_1}{\cos\theta} = J^{-1}KZ_2 = -IZ_2.$$

Then

$$\Delta =< X_4, \tilde{X}_3 >= - < IZ_2, IZ_2 >= -1.$$

The chains $\{X_i\}, \{Y_i\}, \{\tilde{X}_i\}, \{Z_i\}$ of an I-complex subspace with leading vector $X_1 \in U$ w.r.t. the adapted basis (I, J, K) are:

$$\begin{array}{lllllll}
\{X_i\} & = & \{X_1, X_2, X_3, X_4\} & = & \{X_1, -IX_1, Z_2, -IZ_2\} & = & \{X_1, X_2, Z_2, Y_2\} \\
\{Y_i\} & = & \{X_1, Y_2, X_3, Y_4\} & = & \{X_1, -IZ_2, Z_2, IX_1\} & = & \{X_1, Y_2, Z_2, -X_2\} \\
\{\tilde{X}_i\} & = & \{X_1, X_2, \tilde{X}_3, \tilde{X}_4\} & = & \{X_1, -IX_1, IZ_2, Z_2\} & = & \{X_1, X_2, -Y_2, Z_2\} \\
\{Z_i\} & = & \{X_1, Z_2, X_3, Z_4\} & = & \{X_1, Z_2, IZ_2, IX_1\} & = & \{X_1, Z_2, -Y_2, -X_2\}
\end{array} \tag{27}$$

Therefore the set $(\xi, \chi, \eta, \Gamma, \Delta) = (0, 0, 0, 0, -1)$ is an invariant (resp. an intrinsic property) of an I-complex subspace (resp. quaternionic subspace).

In particular for a quaternionic subspace it is $Y_2 = -JX_1$ and $Z_2 = -KX_1$ then

$$\begin{array}{lll}
\{X_i\} & = & \{X_1, -IX_1, -KX_1, -JX_1\} \\
\{Y_i\} & = & \{X_1, -JX_1, -KX_1, IX_1\} \\
\{\tilde{X}_i\} & = & \{X_1, -IX_1, JX_1, -KX_1\} \\
\{Z_i\} & = & \{X_1, -KX_1, JX_1, IX_1\}
\end{array}$$

W.r.t. the canonical bases $\{X_i\}, \{Y_i\}, \{Z_i\}$ the canonical matrices (21) of an I-complex 4-dimensional subspace not totally complex are

$$C_{IJ} = C'_{IK} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad C_{IK} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{28}$$

□

In case the 4-dimensional I-complex subspace (U, I) is totally complex i.e. $U = (U, I, \pi/2)$ we are in a case of double orthogonality. In this case we can always assume $X_2 = Y_2 = Z_2$ (see [21]) and consequently $\xi = \chi = \eta = 1$. From the Definition (3.16) one has $\{X_i\} = \{Y_i\} = \{\tilde{X}_i\} = \{Z_i\}$ and consequently $C_{IJ} = C_{IK} = Id$ as stated in the Corollary (3.19). In this case clearly U is a 2-planes decomposable subspace.

Let denote by $Gr^{\mathbb{R}}_{(I,\theta^{I^\perp})}(4, 4n)$ the subset of 4-dimensional I-complex subspaces of V with $I^\perp$-Kähler angle $\theta^{I^\perp}$ of the Grassmannian $G^{\mathbb{R}}(4, 4n)$.

Theorem 3.33. *Let $(U, I, \theta^{I^\perp}) \in Gr^{\mathbb{R}}_{(I,\theta^{I^\perp})}(4, 4n)$. The pair $(I, \theta^{I^\perp})$ composed by the complex structure $I \in S(\mathcal{Q})$ and the $I^\perp$-Kähler angle $\theta^{I^\perp}$ determines completely the $Sp(n)$-orbit of U in the Grassmannian $Gr^{\mathbb{R}}(4, 4n)$ i.e. the group $Sp(n)$ acts transitively on $Gr^{\mathbb{R}}_{(I,\theta^{I^\perp})}(4, 4n)$. In particular then all totally complex subspaces form one $Sp(n)$-orbit in $Gr^{\mathbb{R}}(4, 4n)$.*

Proof. The proof follows from the Theorem (3.22) and the Proposition (3.21). In fact in case $(U, I, \theta^{I^\perp} \neq \pi/2)$ one bas $(\xi, \chi, \eta, \Delta) = (0, 0, 0, -1)$ and the angles of isoclinicity are $(0, I^\perp, I^\perp)$.

On the other hand, all totally 4 dimensional I-complex subspace are characterized by $(\xi, \chi, \eta, \Delta) = (1, 1, 1, 0)$ and $(\theta^I, \theta^J, \theta^K) = (0, \pi/2, \pi/2)$. □

We terminate the analysis of the 4-dimensional complex subspaces with the

Proposition 3.34. *Let U be a 4-dimensional complex subspace not totally complex. Then $\xi = \chi = \eta = 0$ only w.r.t. an adapted basis.*

Proof. Let consider an I-complex subspace (U, I, θ) and a hypercomplex basis (I', J', K') with $I = \alpha_1 I' + \beta_1 J' + \gamma_1 K'$. Then

$$I' = \alpha_1 I + \dots, \qquad J' = \beta_1 I + \dots, \qquad K' = \gamma_1 I + \dots$$

and square cosine of the angles of isoclinicity $\cos^2 \theta^{I'}, \cos^2 \theta^{J'}, \cos^2 \theta^{K'}$ between the pairs $(U, I'U), (U, J'U), (U, K'U)$. From Proposition (3.35) we have

$$\begin{array}{lcl} \cos(\widehat{U, I'U}) = \cos^2 \theta^{I'} & = & \alpha_1^2 \sin^2 \theta + \cos^2 \theta \\ \cos(\widehat{U, J'U}) = \cos^2 \theta^{J'} & = & \beta_1^2 \sin^2 \theta + \cos^2 \theta \\ \cos(\widehat{U, K'U}) = \cos^2 \theta^{K'} & = & \gamma_1^2 \sin^2 \theta + \cos^2 \theta. \end{array}$$

We can verify that $S = \cos^2\theta^{I'} + \cos^2\theta^{J'} + \cos^2\theta^{K'} = 1 + 2\cos^2\theta = 3\Delta(U)$.

From (20), one has

$$\begin{aligned}\cos\theta^I = 1 \quad = \quad & \alpha_1^2(\alpha_1^2\sin^2\theta + \cos^2\theta) + \beta_1^2(\beta_1^2\sin^2\theta + \cos^2\theta) + \gamma_1^2(\gamma_1^2\sin^2\theta + \cos^2\theta) + \\ & 2 < X_2, Y_2 > \alpha_1\beta_1\cos\theta^{I'}\cos\theta^{J'} + 2 < X_2, Z_2 > \alpha_1\gamma_1\cos\theta^{I'}\cos\theta^{K'} + \\ & 2 < Y_2, Z_2 > \beta_1\gamma_1\cos\theta^{J'}\cos\theta^{K'}\end{aligned}$$

Being

$$(\alpha_1^4 + \beta_1^4 + \gamma_1^4)\sin^2\theta + (\alpha_1^2 + \beta_1^2 + \gamma_1^2)\cos^2\theta = 1 - 2\sin^2\theta(\alpha_1^2\beta_1^2 + \alpha_1^2\gamma_1^2 + \beta_1^2\gamma_1^2)$$

we get

$$\begin{aligned}& 2\alpha_1\beta_1(\alpha_1\beta_1\sin^2\theta - < X_2, Y_2 > \cos\theta^{I'}\cos\theta^{J'}) + \\ & +2\alpha_1\gamma_1(\alpha_1\gamma_1\sin^2\theta - < X_2, Z_2 > \cos\theta^{I'}\cos\theta^{K'}) + \\ & +2\beta_1\gamma_1(\beta_1\gamma_1\sin^2\theta - < Y_2, Z_2 > \cos\theta^{J'}\cos\theta^{K'}) = 0\end{aligned}$$

Then $\xi = \chi = \eta = 0$ if $\sin^2\theta(\alpha_1^2\beta_1^2 + \alpha_1^2\gamma_1^2 + \beta_1^2\gamma_1^2) = 0$ that is either if $\cos\theta = 1$ which implies U quaternionic or if two among $(\alpha_1, \beta_1, \gamma_1)$ are zero i.e. if $I' = \pm I$ or $J' = \pm I$ or $K' = \pm I$. □

Applying the expression (20) for the determination of the angle of isoclinic of the pair (U, AU) for $A \in S(\mathcal{Q})$, we conclude this section with the following

Corollary 3.35. *Given a 4-dimensional I-complex subspace $(U, I, \theta^{I^\perp})$ and the compatible complex structure $A = \alpha I + \beta J + \gamma K$, the cosine of the angle of isoclinicity θ^A between the pair of subspaces U and AU is equal to*

$$\cos\theta^A = \sqrt{\alpha^2 + (1-\alpha^2)\cos^2\theta^{I^\perp}}. \tag{29}$$

Then U is never orthogonal unless it is totally-complex.

3.7 Decomposition of a $2m$-dimensional pure complex subspace

Given a $2m$-dimensional complex subspace $(U, I) \subset V$, let consider the skew-symmetric form $\omega^K : (X, Y) \to < X, KY >$ for $K \in I \perp \cap S(\mathcal{Q})$ and denote by $\omega^K|_U$ its restriction to U.

Proposition 3.36. *Let (U, I) be a $2m$-dimensional I-complex subspace and consider the principal angles of the pair (U, KU). A principal angles $\theta \neq \pi/2$ has multiplicity $4k$; if instead $\theta = \pi/2$ its multiplicity is $2k$.*

Proof. Let (U, I) be a $2m$-dimensional pure I-complex subspace. Consider first the case that U is not totally complex and let U_i be a standard 2-plane of $\omega^K|_U$ with the cosine of the angle of isoclinicity $\cos\theta^K(U_i)$ of the pair $(U_i, KU_i) \neq 0$. Observe that assuming on U_i the orientation induced by a standard basis one has $\cos\theta^K(U_i) = \cos\Theta^K(U_i)$.

Let (X, Z) be an ω^K-standard basis of U_i and consider the I-complexification $\tilde{U}_i = U_i \oplus IU_i \subset U$. From the Proposition (3.29) the generators (X, Z, IX, IZ) are orthonormal which implies that the above direct sum is orthogonal and that $\{Z_i\} = (X, Z, IZ, IX)$ is the ω^K-chain of $\tilde{U}_i$ centered on X. The conclusion follows observing that for the angles of isoclinicity one has $\theta^K(U_i) =< X, KZ >=< IZ, KIX >= \theta^K(IU_i)$.

In case a principal angle $\theta = \pi/2$, let $\bar{U}$ the associated ω^K-standard subspace. From point (5) of the Claim (1.7), it is clearly totally I-complex. Then $\bar{U}$ is 2-planes decomposable and its dimension is necessarily $2k$. □

Theorem 3.37. *Any pure I-complex subspace (U^{2m}, I) admits a decomposition into an* <u>*Hermitian orthogonal*</u> *sum of 4-dimensional pure I-complex subspaces plus, in case the dimension is not multiple of 4, an Hermitian orthogonal (totally) I-complex 2-plane.*

Proof. From Proposition (3.36) one has that each principal angles $\theta \neq \pi/2$ between the pair (U, KU) has multiplicity $4k$. Let denote by $\bar{U}_i$ the ω^K-invariant $4k_i$-dimensional subspaces associated to $\theta_i \neq \pi/2$ and denote by d the sum of their dimensions. Any such subspace is I-complex. In fact by the uniqueness of such invariant ω^K-subspaces and from Proposition (3.36), one has the following decomposition into 4-dimensional I-complex subspaces (U_{ij}, I, θ_i)

$$\bar{U}_i = \bigoplus_{j=1}^{k_i} (U_{ij}, I, \theta_i) = \bigoplus_{j=1}^{k_i} L(X_{ij}, Z_{ij} = \frac{K^{-1} Pr_U^{KU} X_{ij}}{\cos \theta^K}, IX_{ij}, IZ_{ij}),$$

$$X_{ij} \in \bar{U} \cap (\bigoplus_{p=1}^{j-1} U_{ip})^{\perp}.$$

The union of the bases above is an ω^K-standard basis of the I-complex subspace $\bar{U}_i$.

Denoting by $W = \bigoplus \bar{U}_i$, from Claim (1.7), we have that W is a d-dimensional I-complex subspace, it admits a decomposition into 4-dimensional I-complex subspaces and, w.r.t. the orthonormal bases given above, the form $\omega^K|_W$ assumes standard form. The 4-dimensional complex addends of W are Hermitian orthogonal as can be easily seen. In fact, if W_1, W_2 are a pair of such addends, one has that $W_1 \perp W_2 = IW_2$ and $W_1 \perp JW_2 = KW_2$ i.e. $W_1^{\mathbb{H}} \perp W_2^{\mathbb{H}}$. In other words, all different 4-dimensional I-complex addends U_{ij} of the pure I-complex subspace U belong to 8-dimensional quaternionic subspaces orthogonal in pairs.

Finally if $2m - d > 0$, the $(2m - d)$-dimension subspace $W' = W^{\perp} \cap U$ is I-complex being the orthogonal complement to a complex subspace in a complex space (see claim (1.7)). In particular it is totally I-complex since the angle of isoclinicity between the pair (W', KW') is $\pi/2$. It is easy to see that W' is Hermitian orthogonal to W. Furthermore the ω^I standard form restricted to W' determine a decomposition of W' into Hermitian orthogonal 2-dimensional totally I-complex addends. Summing them in pairs the conclusion follows. □

Although the decomposition on the previous Theorem is unique only if all $\cos \theta_i > 0$ have multiplicity 4 and eventually present $\cos \theta_i = 0$ have multiplicity 2, we can state the following corollary whose proof is straightforward.

Corollary 3.38. *To a pure I-complex subspace (U^{4m}, I) we can canonically associate the vector $\boldsymbol{\theta}^{\boldsymbol{I}^\perp} = (\theta_1^{I^\perp}, \dots, \theta_m^{I^\perp})$ where $\theta_i^{I^\perp}$ (ordered in increasing order) are the $I^\perp$-Kähler angles of the Hermitian orthogonal 4-dimensional I-complex subspaces of Theorem (3.37). If $\dim U = 4m+2$, the angle $\theta_{m+1}^{I^\perp} = \pi/2$ is the K-Kähler angle of an Hermitian orthogonal totally complex 2-plane.*

The increasing order of the $I^\perp$-Kähler angles in $\boldsymbol{\theta}$ determines a corresponding order of the relative Hermitian orthogonal 4-dimensional I-complex subspaces.

Definition 3.39. *Let (U, I) be a $2m$-dimensional I-complex subspace. We call the vector $\boldsymbol{\theta}^{I^\perp} = (\theta_1^{I^\perp}, \dots, \theta_{[m/2]}^{I^\perp})$ ($\boldsymbol{\theta} = (\theta_1^{I^\perp}, \dots, \theta_{m/2^{I^\perp}}, \pi/2)$ if m is odd) with $\theta_i^{I^\perp}$ ordered in increasing order, the **$I^\perp$-Kähler multipleangle** of the I-complex $2m$-dimensional subspace (U, I) that we will denote by $(U^{2m}, I, \boldsymbol{\theta}^{\boldsymbol{I}^\perp})$.*

For any leading vector $X_1 = Y_1 = Z_1$, we associate to any 4-dimensional complex subspace the chains $\{X_i\}, \{Y_i\}, \{Z_i\}$ of the Definition (3.15) and given in (27). We recall that in case U is totally complex one has that $\{X_i\} = \{\tilde{X}_i\} = \{Y_i\} = \{\tilde{Y}_i\} = \{Z_i\} = \{\tilde{Z}_i\}$.

Definition 3.40. *The unions of the chains $\{X_i\}$ (resp. $\{Y_i\}, \{Z_i\}$ of the 4-dimensional I-complex addends $(U_{ij}, I, \theta_i^{I^\perp})$ form the triple of the canonical bases of $(U^{2m}, I, \boldsymbol{\theta}^{\boldsymbol{I}^\perp})$.*

Proposition 3.41. *Let $(U^{2m}, I, \boldsymbol{\theta}^{I^\perp})$ a $2m$-dimensional I-complex subspace. If m is even, the canonical matrix C_{IJ} (resp. C_{IK}), w.r.t. the canonical bases, is given by a diagonal block matrices with 4×4-blocks given by the first (resp. the second) of the (28) (plus an order 2 identity block if m is odd).*

Proof. From Theorem (3.37) we have that U admits a standard decomposition into 4-dimensional I-complex subspaces. The addends are Hermitian orthogonal then every orthogonal change of basis preserving such decomposition is represented by a diagonal block matrix with 4×4 blocks. The conclusion follows from Proposition (3.32). $\square$

Let us denote by $Gr^{\mathbb{R}}_{(I, \boldsymbol{\theta}^{I^\perp})}(2m, 4n)$ the set of $2m$-dimensional pure I-complex subspaces in $(V^{4n}, <,>, \mathcal{Q})$ with $I^\perp$-Kähler multipleangle $\boldsymbol{\theta}^{I^\perp}$.

Theorem 3.42. *The group $Sp(n)$ acts transitively on $Gr^{\mathbb{R}}_{(I, \boldsymbol{\theta}^{I^\perp})}(2m, 4n)$ i.e. the pair $(I, \boldsymbol{\theta}^{I^\perp})$ composed by the complex structure $I \in \mathcal{Q}$ and the $I^\perp$-Kähler multipleangle $\boldsymbol{\theta}^{I^\perp}$ of the I-complex subspace U determines completely its $Sp(n)$-orbit in the Grassmannian $Gr^{\mathbb{R}}(2m, 4n)$.*

Proof. The proof follows directly from Proposition (3.37). The Hermitian orthogonality of all the addends of the decomposition of a $2m$-dimensional pure I-complex subspace $(U^{2m}, I, \boldsymbol{\theta}^{I^\perp})$ there stated, allows us to deal separately with each addend since the group $Sp(n)$ preserves such orthogonality. Therefore the canonical matrices w.r.t. the canonical bases given in the Definition (3.40) have the unique form stated in the Proposition (3.41), then from the Theorem (3.3), the pair $(I, \boldsymbol{\theta}^{I^{\perp}})$ determines the $Sp(n)$-orbits of U.

In case $\dim U$ is not multiple of 4, the last addend of the Hermitian orthogonal decomposition stated in Proposition (3.37) is a totally I-complex 2-plane and the conclusion follows from Proposition (3.5). $\square$

If U is A-complex with $A \in S(\mathcal{Q})$ the triple $(\xi, \chi, \eta) \neq (0,0,0)$ w.r.t. an the admissible basis (I, J, K) as stated in the Proposition (3.34) unless $A = \pm I$. Then the canonical matrices never have the form stated in the Proposition (3.41).

3.8 $Sp(n)$-orbit of a Σ-complex subspace

In Proposition (1.14) we stated that a Σ-complex subspace U admits a unique decomposition into Hermitian orthogonal sum of maximal pure complex subspaces by different complex structure. Although in Theorem (3.37) we stated that the decomposition of each I_i-complex $2m$-dimensional addend into 4-dimensional Hermitian orthogonal complex addends (if $m \geq 2$) plus eventually an Hermitian orthogonal totally complex plane (if m is odd) is in general not unique, we have that the $I_i^{\perp}$-Kähler multipleangle is canonically defined. In this case, from Corollary (1.11), we can determine the $Sp(n)$-orbit of U by determining separately the orbit of each complex addend.

Let $U = \bigoplus_{i=1}^{s} (U_i^{2m_i}, I_i, \boldsymbol{\theta}^{I_i^{\perp}})$ where $\boldsymbol{\theta}^{I_i^{\perp}} = (\theta_1^{I_i^{\perp}}, \theta_2^{I_i^{\perp}}, \ldots, \theta_{[m_i/2]}^{I_i^{\perp}})$ is the $I_i^{\perp}$-Kähler multipleangle of the I_i-complex subspace $(U_i, I_i, \boldsymbol{\theta}^{I_i^{\perp}})$ whose elements are the $\theta^{I_i^{\perp}}$-Kähler angles of the 4-dimensional Hermitian orthogonal addends (plus eventually an Hermitian orthogonal totally I_i-complex plane if m_i is odd). Denote by $\mathcal{I} =: (I_1, I_2, \ldots, I_s)$ the vector of the complex structures of the different complex addends (U_i, I_i) ordered as stated in section (1.3) and by $\Theta := (\boldsymbol{\theta}^{I_1^{\perp}}, \ldots, \boldsymbol{\theta}^{I_s^{\perp}})$ the vector whose elements are the respective $I_i^{\perp}$-Kähler multipleangle of each U_i. We can then state the

Theorem 3.43. *The $Sp(n)$-orbit of the Σ-complex subspace U is completely determined by the pair $(\mathcal{I}, \Theta)$.*

Proof. Again the Hermitian orthogonality of the complex 4-dimensional subspaces allows us to consider the orbit of each of them separately. The canonical matrices of each (U_i, I_i), w.r.t. an adapted basis and w.r.t. a canonical basis have the form stated in the Proposition (3.41). For any admissible basis and from Corollary (3.23) the canonical matrices of the Σ-complex subspace U w.r.t. the union of the canonical basis of each I_i-complex subspace have then a unique form. The conclusion follows from the Theorem (3.3). $\square$

In particular this is true for any I_1-complex subspace in which case $\mathcal{I} = I_1$ and $\Theta = \boldsymbol{\theta}^{I_1^{\perp}}$.

References

[1] Afriat S. *Orthogonal and oblique projectors and the characteristics of pair of vector spaces*, Proceeding of the Cambridge Philosophical Society **53** (1957), 800-816.

[2] D.V. Alekseevsky, S. Marchiafava, *A report on quaternionic-like structures on a manifold*, Proceedings of the International Workshop on Differential Geometry and its Applications (Bucharest, 1993). Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. 55 (1993), no. 3-4, 9-34.

[3] M.Bruni: *Su alcune proprietà di geometria euclidea ed Hermitiana in uno spazio vettoriale quaternionale*, Ann. Mat. Pura Appl. **72** (1965), 59-77.

[4] M.Bruni: *Misure angolari in uno spazio vettoriale quaternionale*, Rend. di Mat. e Appl. **25** (1966), 394-401.

[5] M.Bruni: *Misure euclidee, hermitiane, simplettiche e potenze esterne di uno spazio vettoriale quaternionale*, Ann. Mat. Pura Appl. **78** (1971), 71-97.

[6] A. Galantai, Cs. J. Hegedus, *Jordan's principal angles in complex vector spaces*. Numer. Linear Algebra Appl. 13, (2006), 589-598

[7] A.Gray, *A note on manifold whose holonomy group is a subgroup of* $Sp(n) \cdot Sp(1)$ Mith. Math. J. 16 (1969), 125-128.

[8] S. Gutmann, L. Shepp, *Orthogonal bases for two subspaces with all mutual angles acute*. Indiana University Mathematics J. 27 no.1 (1978), 79-90.

[9] H. J. Kang, H. Tasaki, *Integral geometry of real surfaces in complex projective spaces*. Tsukuba J. Math. 25 (2001), no. 1, 155-164.

[10] G.B.Rizza, *On the geometry of a pair of oriented planes*. Riv. Mat. Univ. Parma (6) 4 (2001), 217-228.

[11] G.B.Rizza, *Deviazione caratteristica e proprietà locali delle* $2q$*-faccette di una* V_{2n} *a struttura complessa*. Rend. Acc. Naz. XL (1959),191-205.

[12] S. M. Salamon, *Quaternion-Kähler geometry*. Surveys in differential geometry: essays on Einstein manifolds, Surv. Differ. Geom., VI, Int. Press, Boston, MA (1999), 83-121.

[13] K. Sharnhorst, *Angles in Complex Vector Spaces*. Acta Applicandae Matematicae 69 (2001), 95-103.

[14] Sheng Jiang, *Angles between Eulidean subspaces*. Geometriae Dediata 63 (1996), 113-121.

[15] H. Tasaki, *Generalization of Kähler angle and integral geometry in complex projective spaces*, Steps in Differential Geometry, Proceedings of Colloquium on Differential Geometry, Debrecen 2000, published by Inst. Math. Inform., Debrecen (2001), 349-361.

[16] M.Vaccaro: *Subspaces of a para-quaternionic Hermitian vector space*, Int. J. Geom. Methods Mod. Phys. (7) **8** (2011), 1487-1506.

[17] M.Vaccaro: *Basics of linear para-quaternionic geometry I: Hermitian para-type structure on a real vector space*, Bull. Soc. Sci. Lettres Łódź Sér. Rech. Déform (1) **61** (2011), 17-30.

[18] M.Vaccaro: *Basics of linear para-quaternionic geometry II: Decomposition of a generic subspace of a para-quaternionic Hermitian vector space*, Bull. Soc. Sci. Lettres Łódź Sér. Rech. Déform (2) **61** (2011), 15-32.

[19] M.Vaccaro, *Orbits in the real Grassmannian of 2-planes under the action of the groups $Sp(n)$ and $Sp(n) \cdot Sp(1)$*, Bull. Soc. Sci. Lettres Łódź Sér. Rech. Déform. 65 (2015), no.3, December 2015. 129-148.

[20] M.Vaccaro, *$Sp(n)$-orbits in the real Grassmannian $G^{\mathbb{R}}(k, 4n)$* Soon on arXiv.

[21] M.Vaccaro, *Orbits of isoclinic subspaces in the real Grassmannians*, arXiv 4049981, November 2021.

[22] P.Zhu, A.V. Knyazev: *Angles between subspaces and their tangents*, Journal of Numerical Mathematics, 2013

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$Sp(n)$-orbity w Grassmannianach podprzestrzeni zespolonych i Σ-złożonych hermitowskiej kwaternionowej przestrzeni wektorowej

S t r e s z c z e n i e

Określamy niezmienniki charakteryzujące orbity $Sp(n)$ w rzeczywistym Grassmannanie $Gr^{\mathbb{R}}(2k, 4n)$ złożonej $2k$-wymiarowej i Σ-złożonej podprzestrzeni $4n$ - wymiarowa hermitowska kwaternionowa przestrzeń wektorowa. Podprzestrzeń Σ-złożona jest sumą ortogonalną złożonych podprzestrzeni według różnej, aż do znaku, zgodnej struktury złożonej. Wynik otrzymujemy rozpatrując dwie główne cechy takich podprzestrzeni. Po pierwsze, każda taka podprzestrzeń dopuszcza rozkład na hermitowską sumę ortogonalną 4-wymiarowych złożonych dodatków plus 2-wymiarową całkowicie złożoną podprzestrzeń, jeśli k jest nieparzyste, co oznacza, że kwaternionizacja dodatków jest ortogonalna w parach. Po drugie, każdy 4-wymiarowy dodatek złożony U jest podprzestrzenią izokliniczną, tj. kąty główne pary (U, AU) są takie same dla każdej zgodnej struktury złożonej A. Używając tych własności

określamy pełny zbiór niezmienników charakteryzujących $Sp(n)$-orbitę takich podprzestrzeni w $Gr^{\mathbb{R}}(2k, 4n)$.

Słowa kluczowe: Hermitowska struktura hiperzespolona, hermitowska struktura kwaternionowa, zespolone podprzestrzenie, kąty główne, kąty Kählera