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ON THE SPECTRA OF A QUATERNION SINGULAR INTEGRAL OPERATOR AND ITS COMPONENTS ON SPATIAL SURFACES

Summary

We study point spectra of a quaternion singular integral operator and real singular integral operators which are its components; the operators act on the Hölder space $\mathbf{H}^{\alpha}(\Gamma)$, $\alpha \in (0,1)$, where Γ is a closed quadrable surface.

Keywords and phrases: quaternion, differentiable function, hyperholomorphic function, singular integral Cauchy operator, spectrum

1. Introduction

We used in this work the method of study of the point spectrum of integral operators, which resulted from discussions with prof. Mikhael Shapiro and prof. Vladimir Kutrunov in Mexico City on 2006.

In the work [1] we studied point spectra of two integral operators, generated by the Cauchy singular integral on a closed Lyapunov curve in the complex plane $\mathbb{C}$. This method was further developed in [2], where we investigate the point spectra of the reduced singular integral Cauchy operator and two real integral operators which are its components on a closed regular curve (i.e. the measure of its intersection with a disk does not exceed a constant multiplied by the radius of the disk). In the present work we consider the same problem for a quaternion singular integral Cauchy operator and its components on a closed quadrable surface in $\mathbb{R}^3$.

2. Quaternion holomorphic functions and surface integrals

Let $\mathbb{H}(\mathbb{C})$ be the associative algebra of complex quaternions

$$a = \sum_{k=0}^{3} a_k \boldsymbol{i}_k,$$

where $\{a_k\}_{k=0}^{3} \subset \mathbb{C}$, $\boldsymbol{i}_0 = 1$ and $\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3$ be the imaginary quaternion units with the multiplication rule $\boldsymbol{i}_1^2 = \boldsymbol{i}_2^2 = \boldsymbol{i}_3^2 = \boldsymbol{i}_1\boldsymbol{i}_2\boldsymbol{i}_3 = -1$. The module of the quaternion a is defined by the formula

$$|a| := \sqrt{\sum_{k=0}^{3} |a_k|^2}.$$

For complex quaternions the next relation is true:

$$|ab| \leqslant \sqrt{2}\,|a|\,|b| \tag{1}$$

(see Lemma 2.1 of [3]).

For $\{z_k\}_{k=1}^{3} \subset \mathbb{R}$ consider vector quaternions $z := z_1\boldsymbol{i}_1 + z_2\boldsymbol{i}_2 + z_3\boldsymbol{i}_3$ as points of the Euclidean space $\mathbb{R}^3$ with the basis $\{\boldsymbol{i_k}\}_{k=1}^{3}$. Let Ω be a domain of $\mathbb{R}^3$. For functions $f : \Omega \to \mathbb{H}(\mathbb{C})$ having first order partial derivatives consider quaternion differential operators

$$D_l[f] := \sum_{k=1}^{3} \boldsymbol{i}_k \frac{\partial f}{\partial z_k},$$

$$D_r[f] := \sum_{k=1}^{3} \frac{\partial f}{\partial z_k} \boldsymbol{i}_k,$$

factorizing the Laplace operator

$$\Delta_{\mathbb{R}^3} := \frac{\partial^2}{\partial z_1^2} + \frac{\partial^2}{\partial z_2^2} + \frac{\partial^2}{\partial z_3^2} = D_l \circ (-D_l) = D_r \circ (-D_r) \tag{2}$$

like the complex factorization of the two-dimensional Laplace operator

$$\Delta_{\mathbb{R}^2} := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \circ \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right).$$

Definition 2.1. Function $f := f_0\boldsymbol{i_0} + f_1\boldsymbol{i_1} + f_2\boldsymbol{i_2} + f_3\boldsymbol{i_3}$ is called **left $\mathbb{H}$-differentiable** or **right $\mathbb{H}$-differentiable** in a point $z^{(0)} \in \mathbb{R}^3$, when its components f_0, f_1, f_2, f_3 are $\mathbb{R}^3$-differentiable functions in $z^{(0)}$ and the condition

$$D_l[f](z^{(0)}) = 0 \tag{3}$$

or

$$D_r[f](z^{(0)}) = 0.$$

holds true respectively.

Defined in such a way the notion of the $\mathbb{H}$-differentiability of a quaternion function is the exact analogue of the $\mathbb{C}$-differentiability of a complex function (see [4]) $f(\zeta) = u(x,y)+v(x,y)\boldsymbol{i}$ at the point $\zeta_0 = x_0+y_0\boldsymbol{i}$, that is equivalent to $\mathbb{R}^2$-differentiability of the components $u(x,y)$, $v(x,y)$ at the point (x_0,y_0) and fulfillment of the condition

$$\frac{\partial f(\zeta_0)}{\partial x} + \frac{\partial f(\zeta_0)}{\partial y}\boldsymbol{i} = 0.$$

It is known (see [4]) that $\mathbb{C}$-differentiability of a complex function is equivalent to existence of its derivative. In hypercomplex analysis there is no such the equivalence. Particulary in quaternion analysis only linear functions of the special form have derivatives (see [5]). Operator D_l is called the Dirac operator or Moisil-Theodoresco operator, and the equality (3) is equivalent to the Moisil-Theodoresco system [6]

Definition 2.2. A function is called **left-hyperholomorphic** or **right-hyperholomorphic** in a domain Ω when it is left $\mathbb{H}$-differentiable or right $\mathbb{H}$-differentiable in each point of the domain Ω respectively.

Definition 2.3. A function is called **hyperholomorphic** in a domain Ω when it is left-hyperholomorphic and at the same time right-hyperholomorphic in the domain Ω.

The fundamental solution of the the Laplace operator in the space $\mathbb{R}^3$ has the following form:

$$\theta(z) = -\frac{1}{4\pi|z|}$$

(i.e. $\Delta_{\mathbb{R}^3}[\theta](z) = \delta(z)$, where $\delta(z)$ is the Dirac delta-function).

Due to the factorizations (2), the **quaternion Cauchy kernel**

$$K(z) := -D_l[\theta](z) = -D_r[\theta](z) = -\frac{z}{4\pi|z|^3}$$

is a fundamental solution of operators D_l, D_r and therefore it is a hyperholomorphic function in the domain $\mathbb{C}\setminus\{0\}$.

Definition 2.4 ([7, 8])**.** A surface $\Gamma \subset \mathbb{R}^3$ is an image of a closed set $G \subset \mathbb{R}^2$ under a homeomorphic mapping $\varphi : G \to \mathbb{R}^3$

$$z = \varphi(u,v) := z_1(u,v)\boldsymbol{i}_1 + z_2(u,v)\boldsymbol{i}_2 + z_3(u,v))\boldsymbol{i}_3\,, (u,v) \in G,$$

such that Jacobians

$$A(u,v) := \frac{\partial z_2}{\partial u}\frac{\partial z_3}{\partial v} - \frac{\partial z_2}{\partial v}\frac{\partial z_3}{\partial u},\ B(u,v) := \frac{\partial z_3}{\partial u}\frac{\partial z_1}{\partial v} - \frac{\partial z_3}{\partial v}\frac{\partial z_1}{\partial u},$$

$$C(u,v) := \frac{\partial z_1}{\partial u}\frac{\partial z_2}{\partial v} - \frac{\partial z_1}{\partial v}\frac{\partial z_2}{\partial u}$$

exist almost everywhere on the set G and summable on G.

It is known that the vector $\boldsymbol{n}(u,v) := A(u,v)\boldsymbol{i}_1+B(u,v)\boldsymbol{i}_2+C(u,v)\boldsymbol{i}_3$ is a normal vector to the surface Γ at the point $z \in \Gamma$.

The area of the surface Γ is calculated by the formula

$$\mathcal{L}(\Gamma) = \iint\limits_{G} |\boldsymbol{n}(u,v)|dudv,$$

where the integral is understood in Lebesgue sense.

A surface Γ is called **quadrable** (see [7]), if $\mathcal{L}(\Gamma) < +\infty$.

Let $\Gamma \subset \mathbb{R}^3$ be an image of a sphere $S \subset \mathbb{R}^3$ in a such homeomorphic mapping $\psi : S \to \mathbb{R}^3$ that the image of a great circle γ on the sphere S is a close Jordan rectifiable curve $\widetilde{\gamma}$ on the set Γ.

The sphere S is the union of two half-spheres S_1, S_2 with common edge γ.

It is ease to see that there exist continuously differentiable mappings $\varphi_1 : B \to S_1$, $\varphi_2 : B \to S_2$ of the unit disk $B := \{(u,v) \in \mathbb{R}^2 : u^2 + v^2 \leqslant 1\}$. So the set Γ is the union of two sets $\Gamma_1 = \psi(\varphi_1(B))$, $\Gamma_2 = \psi(\varphi_2(B))$ with the intersection $\widetilde{\gamma} = \psi(\varphi_1(\partial B)) = \psi(\varphi_2(\partial B))$.

Definition 2.5. A set Γ is called a **closed surface** if there exist a such homeomorphic mapping $\psi : S \to \mathbb{R}^3$ that the sets Γ_1, Γ_2 are surfaces in the sense of Definition 2.4 and orientation of the circle ∂B induces two mutually opposite orientations of the curve $\widetilde{\gamma}$ under mappings $\psi \circ \varphi_1$ and $\psi \circ \varphi_2$ respectively.

Let $\Gamma^\varepsilon := \{z \in \mathbb{R}^3 : \rho(z,\Gamma) \leqslant \varepsilon\}$ (ρ denotes the Euclidean distance) be the closed ε-neighborhood of the surface Γ, $V(\Gamma^\varepsilon)$ be the space Lebesgue measure of the set Γ^ε and $\mathcal{M}^*(\Gamma) := \overline{\lim\limits_{\varepsilon\to 0}} \frac{V(\Gamma^\varepsilon)}{2\varepsilon}$ be the two-dimensional upper Minkowski content (see [9, p. 79]) of the surface Γ.

For functions $f : \Gamma \to \mathbb{H}(\mathbb{C})$, $g : \Gamma \to \mathbb{H}(\mathbb{C})$ in the case of non-closed quadrable surface Γ the **quaternion surface integral** is defined by the formula

$$\iint\limits_{\Gamma} f(z)\,\sigma(z)\,g(z) := \iint\limits_{G} f(\varphi(u,v))\boldsymbol{n}(u,v)g(\varphi(u,v))du\,dv$$

where $\sigma(z) := dz_2dz_3\boldsymbol{i}_1 + dz_3dz_1\boldsymbol{i}_2 + dz_1dz_2\boldsymbol{i}_3$ is a formal representation of the differential form σ and in the case of a closed surface the surface integral is defined by the formula

$$\iint\limits_{\Gamma} f(z)\,\sigma(z)\,g(z) := \iint\limits_{\Gamma_1} f(z)\,\sigma(z)\,g(z) + \iint\limits_{\Gamma_2} f(z)\,\sigma(z)\,g(z).$$

A close surface Γ is called a **positively oriented surface** if the normal vector $\boldsymbol{n}(u,v)$ is directed outside the bounded domain with the boundary Γ at every point of the surface Γ, where it exists.

Theorem 2.6 ([8]). *Let $\mathbb{R}^3 \supset \Omega$ be a bounded domain with a quadrable closed boundary Γ, for which*

$$\mathcal{M}^*(\Gamma) < +\infty,$$

let function $f : \overline{\Omega} \to \mathbb{H}(\mathbb{C})$ be right-hyperholomorphic in Ω and continuous in the closure $\overline{\Omega}$ and let function $g : \overline{\Omega} \to \mathbb{H}(\mathbb{C})$ be left-hyperholomorphic in Ω and continuous in $\overline{\Omega}$. Then

$$\iint\limits_{\Gamma} f(z)\,\sigma(z)\,g(z) = 0.$$

Theorem 2.7. *Let the boundary Γ of a bounded domain Ω^+ be a positively oriented closed quadrable surface and let f be a left-hyperholomorphic function in the domain Ω^+ and continuous on its closure $\overline{\Omega^+}$. Then the Cauchy integral formula*

$$\iint\limits_{\Gamma} K(\zeta - z)\,\sigma(\zeta)\,f(\zeta) = \begin{cases} f(z), & z \in \Omega^+, \\ 0, & z \in \Omega^- \end{cases}$$

holds true.

Proof. Let $\gamma_{z,\rho}$ be the positively oriented sphere of the radius ρ with the center $z = z_1\boldsymbol{i}_1 + z_2\boldsymbol{i}_2 + z_3\boldsymbol{i}_3 \in \Omega^+$ assigned by the equations set

$$\begin{cases} \zeta_1 = z_1 + \rho\sin\theta\cos\varphi, \\ \zeta_2 = z_2 + \rho\sin\theta\sin\varphi, \\ \zeta_3 = z_3 + \rho\cos\theta, \end{cases} \quad \begin{array}{c} 0 \leqslant \theta \leqslant \pi, \\ 0 \leqslant \varphi \leqslant 2\pi. \end{array}$$

It follows from Theorem 2.6 that

$$\iint\limits_{\Gamma} K(\zeta - z)\,\sigma(\zeta)\,f(\zeta) = \iint\limits_{\gamma_{z,\rho}} K(\zeta - z)\,\sigma(\zeta)\,f(\zeta). \tag{4}$$

Due to the continuity of the function f we have

$$\forall\varepsilon > 0\,\exists\delta(\varepsilon) > 0 : |\zeta - z| < \delta(\varepsilon) \Rightarrow |f(\zeta) - f(z)| < \varepsilon.$$

By using the equality

$$\iint\limits_{\gamma_{z,\rho}} K(\zeta - z)\,\sigma(\zeta) = 1 \tag{5}$$

(it is a partial case of the formula (4.13) of the work [10]) and the inequality (1), it follows from the inequality $|\zeta - z| < \delta(\varepsilon)$ that

$$\left| \iint\limits_{\gamma_{z,\rho}} K(\zeta - z)\,\sigma(\zeta)\,f(\zeta) - f(z) \right| =$$

$$= \left| \iint\limits_{\gamma_{z,\rho}} K(\zeta - z)\,\sigma(\zeta)\,f(\zeta) - \iint\limits_{\gamma_{z,\rho}} K(\zeta - z)\,\sigma(\zeta)\,f(z) \right| =$$

$$= \left| \iint\limits_{\gamma_{z,\rho}} K(\zeta - z)\,\sigma(\zeta)\,(f(\zeta) - f(z)) \right| \leqslant \sqrt{2} \iint\limits_{\gamma_{z,\rho}} |K(\zeta - z)|\,|\sigma(\zeta)|\,|f(\zeta) - f(z)| \leqslant$$

$$\leqslant \sqrt{2}\varepsilon \iint\limits_{\gamma_{z,\rho}} |K(\zeta - z)|\,|\sigma(\zeta)| = 2\sqrt{2}\varepsilon \iint\limits_{\gamma^+_{z,\rho}} |K(\zeta - z)|\,|\sigma(\zeta)| =$$

$$= 2\sqrt{2}\varepsilon \int\limits_0^{2\pi} d\varphi \int\limits_0^{\frac{\pi}{2}} \frac{1}{4\pi} \sin\theta d\theta = \sqrt{2}\varepsilon,$$

where $\gamma^+_{z,\rho}$ is the upper half-sphere of the sphere $\gamma_{z,\rho}$ (with $0 \leqslant \theta \leqslant \frac{\pi}{2}$). Therefore

$$\iint\limits_{\gamma_{z,\rho}} K(\zeta - z)\,\sigma(\zeta)\,f(\zeta) = f(z) + \alpha(\rho), \tag{6}$$

where the left side integral does not depend on ρ and $\alpha(\rho) \to 0$ as $\rho \to 0$. By the relations (4), (6) the case $z \in \Omega^+$ of the Theorem is proved.

In the case $z \in \Omega^-$ the Cauchy kernel $K(\zeta - z)$ is a hyperholomorphic function of the variable ζ in the domain Ω^+ and continuous on its closure $\overline{\Omega^+}$. So by Theorem 2.6 we have

$$\iint\limits_{\Gamma} K(\zeta - z)\,\sigma(\zeta)\,f(\zeta) = 0.$$

Theorem is proved. □

Theorem 2.8. *Let the boundary Γ of a non-bounded domain Ω^- be a positively oriented closed quadrable surface, let f be a left-hyperholomorphic function in the domain Ω^- and continuous on its closure $\overline{\Omega^-}$ and let the limit*

$$f(\infty) := \lim_{z\to\infty} f(z)$$

exists and is finite. Then the Cauchy integral formula

$$\iint\limits_{\Gamma} K(\zeta - z)\,\sigma\,f(\zeta) = \begin{cases} f(\infty) - f(z), & z \in \Omega^-, \\ f(\infty), & z \in \Omega^+, \end{cases}$$

holds true.

Proof. Let $\gamma_{z,\rho}$ be the positively oriented sphere of the radius ρ with the center $z = z_1\boldsymbol{i}_1 + z_2\boldsymbol{i}_2 + z_3\boldsymbol{i}_3 \in \Omega^-$.

It follows from Theorem 2.7 that for every ρ that is so large that $|\zeta - z| < \rho$ for all $\zeta \in \Gamma$

$$\iint\limits_{\Gamma} K(\zeta - z)\,\sigma(\zeta)\,f(\zeta) = \iint\limits_{\gamma_{z,\rho}} K(\zeta - z)\,\sigma(\zeta)\,f(\zeta) - f(z).$$

Due to the definition of $f(\infty)$ we have

$$\forall\varepsilon > 0\,\exists M(\varepsilon) > 0 : |\zeta - z| > M(\varepsilon) \Rightarrow |f(\zeta) - f(\infty)| < \varepsilon.$$

Then by using the equality (5) and the inequality (1), it follows from the inequality $\rho > M(\varepsilon)$ that

$$\left|\iint\limits_{\gamma_{z,\rho}} K(\zeta - z)\,\sigma(\zeta)\,f(\zeta) - f(\infty)\right| =$$

$$= \left|\iint\limits_{\gamma_{z,\rho}} K(\zeta - z)\,\sigma(\zeta)\,f(\zeta) - \iint\limits_{\gamma_{z,\rho}} K(\zeta - z)\,\sigma(\zeta)\,f(\infty)\right| =$$

$$= \left|\iint\limits_{\gamma_{z,\rho}} K(\zeta - z)\,\sigma(\zeta)\,(f(\zeta) - f(\infty))\right| \leqslant \sqrt{2}\iint\limits_{\gamma_{z,\rho}} |K(\zeta - z)|\,|\sigma(\zeta)|\,|f(\zeta) - f(\infty)| <$$

$$< \sqrt{2}\varepsilon \iint\limits_{\gamma_{z,\rho}} |K(\zeta - z)|\,|\sigma(\zeta)| = \sqrt{2}\varepsilon.$$

Consequently

$$\iint\limits_{\gamma_{z,\rho}} K(\zeta - z)\,\sigma(\zeta)\,f(\zeta) = f(\infty)$$

and Theorem is proved in the case $z \in \Omega^-$.

In the case $z \in \Omega^+$ we have for every ρ that is so large that $|\zeta - z| < \rho$ for all $\zeta \in \Gamma$

$$\iint\limits_{\Gamma} K(\zeta - z)\,\sigma(\zeta)\,f(\zeta) = \iint\limits_{\gamma_{z,\rho}} K(\zeta - z)\,\sigma(\zeta)\,f(\zeta) = f(\infty)$$

and Theorem is proved. □

3. Quaternion singular integral Cauchy operator and its components

Let $f : \Gamma \to \mathbb{H}(\mathbb{C})$ be a continuous function from a Hölder space $H^\alpha(\Gamma)$, $\alpha \in (0,1)$ (i.e. exists a positive constant L, such as for any points ζ_1, ζ_2 of the surface Γ the inequality

$$|f(\zeta_1) - f(\zeta_2)| \leqslant L|\zeta_1 - \zeta_2|^\alpha$$

holds true).

A norm in the space $H^\alpha(\Gamma)$ is defined by the formula:

$$\|f\|_\alpha := \max_{\zeta\in\Gamma} |f(\zeta)| + \sup_{\{\zeta_1;\zeta_2\}\subset\Gamma;\zeta_1\neq\zeta_2} \frac{|f(\zeta_1) - f(\zeta_2)|}{|\zeta_1 - \zeta_2|^\alpha}.$$

Let Γ be a quadrable surface and let

$$\mathbf{\Phi}[f](z) := \iint\limits_{\Gamma} K(\zeta - z)\,\sigma(\zeta)\,f(\zeta), \quad z \in \mathbb{R}^3 \setminus \Gamma, \tag{7}$$

be the quaternion Cauchy-type integral.

Theorem 3.1. *The Cauchy-type integral (7) is a hyperholomorphic function in the domain* $\mathbb{R}^3 \setminus \Gamma$.

Proof. By using representations $K(\zeta - z) = -\dfrac{\sum\limits_{k=1}^{3}(\zeta_k - z_k)\boldsymbol{i}_k}{4\pi\left(\sum\limits_{p=1}^{3}(\zeta_p - z_p)^2\right)^{\frac{3}{2}}}$,

$\sigma(\zeta) = d\zeta_2 d\zeta_3 \boldsymbol{i}_1 + d\zeta_3 d\zeta_1 \boldsymbol{i}_2 + d\zeta_1 d\zeta_2 \boldsymbol{i}_3$, $f(\zeta) = \sum\limits_{l=0}^{3} f_l(\zeta)\boldsymbol{i}_l$ let us represent the Cauchy-type integral (7) by components

$$\mathbf{\Phi}[f](z) = -\frac{1}{4\pi}\sum_{k=1}^{3}\sum_{l=0}^{3}\iint\limits_{\Gamma}\frac{(\zeta_k - z_k)f_l(\zeta)}{\left(\sum\limits_{p=1}^{3}(\zeta_p - z_p)^2\right)^{\frac{3}{2}}}d\zeta_2 d\zeta_3 \boldsymbol{i}_k\boldsymbol{i}_1\boldsymbol{i}_l -$$

$$-\frac{1}{4\pi}\sum_{k=1}^{3}\sum_{l=0}^{3}\iint\limits_{\Gamma}\frac{(\zeta_k - z_k)f_l(\zeta)}{\left(\sum\limits_{p=1}^{3}(\zeta_p - z_p)^2\right)^{\frac{3}{2}}}d\zeta_3 d\zeta_1 \boldsymbol{i}_k\boldsymbol{i}_2\boldsymbol{i}_l -$$

$$-\frac{1}{4\pi}\sum_{k=1}^{3}\sum_{l=0}^{3}\iint\limits_{\Gamma}\frac{(\zeta_k - z_k)f_l(\zeta)}{\left(\sum\limits_{p=1}^{3}(\zeta_p - z_p)^2\right)^{\frac{3}{2}}}d\zeta_1 d\zeta_2 \boldsymbol{i}_k\boldsymbol{i}_3\boldsymbol{i}_l.$$

Then

$$D_r[\mathbf{\Phi}](z) = -\frac{1}{4\pi}\sum_{q=1}^{3}\sum_{k=1}^{3}\sum_{l=0}^{3}\frac{\partial}{\partial z_q}\iint\limits_{\Gamma}\frac{(\zeta_k - z_k)f_l(\zeta)}{\left(\sum\limits_{p=1}^{3}(\zeta_p - z_p)^2\right)^{\frac{3}{2}}}d\zeta_2 d\zeta_3 \boldsymbol{i}_k\boldsymbol{i}_1\boldsymbol{i}_l\boldsymbol{i}_q -$$

$$-\frac{1}{4\pi}\sum_{q=1}^{3}\sum_{k=1}^{3}\sum_{l=0}^{3}\frac{\partial}{\partial z_q}\iint\limits_{\Gamma}\frac{(\zeta_k - z_k)f_l(\zeta)}{\left(\sum\limits_{p=1}^{3}(\zeta_p - z_p)^2\right)^{\frac{3}{2}}}d\zeta_3 d\zeta_1 \boldsymbol{i}_k\boldsymbol{i}_2\boldsymbol{i}_l\boldsymbol{i}_q -$$

$$-\frac{1}{4\pi}\sum_{q=1}^{3}\sum_{k=1}^{3}\sum_{l=0}^{3}\frac{\partial}{\partial z_q}\iint\limits_{\Gamma}\frac{(\zeta_k-z_k)f_l(\zeta)}{\left(\sum\limits_{p=1}^{3}(\zeta_p-z_p)^2\right)^{\frac{3}{2}}}d\zeta_1 d\zeta_2 \boldsymbol{i}_k\boldsymbol{i}_3\boldsymbol{i}_l\boldsymbol{i}_q. \tag{8}$$

For example by using the Lagrange theorem we obtain

$$\frac{\partial}{\partial z_1}\iint\limits_{\Gamma}\frac{(\zeta_1-z_1)f_l(\zeta)}{\left(\sum\limits_{p=1}^{3}(\zeta_p-z_p)^2\right)^{\frac{3}{2}}}d\zeta_2 d\zeta_3 =$$

$$=\lim_{h\to 0}\frac{1}{h}\left(\iint\limits_{\Gamma}\frac{(\zeta_1-(z_1+h))f_l(\zeta)}{\left((\zeta_1-(z_1+h))^2+\sum\limits_{p=2}^{3}(\zeta_p-z_p)^2\right)^{\frac{3}{2}}}d\zeta_2 d\zeta_3 -\right.$$

$$\left.-\iint\limits_{\Gamma}\frac{(\zeta_1-z_1)f_l(\zeta)}{\left(\sum\limits_{p=1}^{3}(\zeta_p-z_p)^2\right)^{\frac{3}{2}}}d\zeta_2 d\zeta_3\right) =$$

$$=\lim_{h\to 0}\iint\limits_{\Gamma}\frac{2(\zeta_1-(z_1+\theta h)))^2-\sum\limits_{p=2}^{3}(\zeta_p-z_p)^2}{\left((\zeta_1-(z_1+\theta h))^2+\sum\limits_{p=2}^{3}(\zeta_p-z_p)^2\right)^{\frac{5}{2}}}f_l(\zeta)d\zeta_2 d\zeta_3 =$$

$$=\lim_{h\to 0}\iint\limits_{G}\Phi(u,v,\theta h)f_l(\zeta(\varphi(u,v)))A(u,v)dudv,$$

where

$$\Phi(u,v,\theta h):=\frac{2(\zeta_1(\varphi(u,v))-(z_1+\theta h)))^2-\sum\limits_{p=2}^{3}(\zeta_p(\varphi(u,v))-z_p)^2}{\left((\zeta_1(\varphi(u,v))-(z_1+\theta h))^2+\sum\limits_{p=2}^{3}(\zeta_p(\varphi(u,v))-z_p)^2\right)^{\frac{5}{2}}}$$

and $0<\theta<1$.

There exists such $h_1>0$ that $\Phi(u,v,\theta h)$ is a continuous function of the variable $(u,v,\theta h)$ belonging to the closed bounded set $G\times[0;h_1]\subset\mathbb{R}^3$. That is why it is a uniformly continuous function. And therefore

$$\forall\varepsilon>0\,\exists\delta(\varepsilon)>0\,\forall(u,v)\in G: 0\leqslant h<\delta(\varepsilon)\Rightarrow$$

$$|\Phi(u,v,\theta h)-\Phi(u,v,0)|<\varepsilon.$$

Then

$$\left|\iint\limits_G \Phi(u,v,\theta h)f_l(\zeta(\varphi(u,v)))A(u,v)dudv-\iint\limits_G \Phi(u,v,0)f_l(\zeta(\varphi(u,v)))A(u,v)dudv\right|$$

$$<\iint\limits_G |\Phi(u,v,\theta h)-\Phi(u,v,0)|\,|f_l(\zeta(\varphi(u,v)))A(u,v)|dudv<$$

$$<\varepsilon\iint\limits_G |f_l(\zeta(\varphi(u,v)))A(u,v)|dudv.$$

Since the last integral is finite due to the summability of the integrand, then

$$\frac{\partial}{\partial z_1}\iint\limits_\Gamma \frac{(\zeta_1-z_1)f_l(\zeta)}{\left(\sum\limits_{p=1}^{3}(\zeta_p-z_p)^2\right)^{\frac{3}{2}}}d\zeta_2d\zeta_3=\iint\limits_G \Phi(u,v,0)f_l(\zeta(\varphi(u,v)))A(u,v)dudv=$$

$$=\iint\limits_\Gamma \frac{3(\zeta_1-z_1)^2-\sum\limits_{p=1}^{3}(\zeta_p-z_p)^2}{\left(\sum\limits_{p=1}^{3}(\zeta_p-z_p)^2\right)^{\frac{5}{2}}}f_l(\zeta)d\zeta_2d\zeta_3.$$

The same equalities for another components of $D_r[\mathbf{\Phi}](z)$ are proved analogously. In particular

$$\frac{\partial}{\partial z_1}\iint\limits_\Gamma \frac{(\zeta_2-z_2)f_l(\zeta)}{\left(\sum\limits_{p=1}^{3}(\zeta_p-z_p)^2\right)^{\frac{3}{2}}}d\zeta_2d\zeta_3=\iint\limits_\Gamma \frac{3(\zeta_2-z_2)^2}{\left(\sum\limits_{p=1}^{3}(\zeta_p-z_p)^2\right)^{\frac{5}{2}}}f_l(\zeta)d\zeta_2d\zeta_3.$$

Substituting all such equalities into the formula (8), we get after simplification that

$$D_r[\mathbf{\Phi}](z)=\iint\limits_\Gamma D_r[K](\zeta-z)\,\sigma(\zeta)\,f(\zeta).$$

Since the Cauchy kernel $K(\zeta-z)$ is a hyperholomorphic function of the variable z then $D_r[K](\zeta-z)=0$ and hence $D_r[\mathbf{\Phi}](z)=0$. The equality $D_l[\mathbf{\Phi}](z)=0$ is proved similarly. Theorem is proved. $\square$

Let $\mathbf{\Phi}^+[f](z)$, $z\in\overline{\Omega^+}$, be the continuous extension of $\mathbf{\Phi}[f]$ onto closure $\overline{\Omega^+}$ of the domain Ω^+ (if it exists), $\mathbf{\Phi}^-[f](z)$, $z\in\overline{\Omega^-}$, be the continuous extension of $\mathbf{\Phi}[f]$ onto closure $\overline{\Omega^-}$ of the domain Ω^- (if it exists).

Theorem 3.2 ([11]). *Let Ω be a Jordan domain with the boundary Γ having a finite Hausdorff measure of dimension 2, let $f:\Gamma\to\mathbb{H}(\mathbb{C})$ be a continuous function, and* *let the integral*

$$\mathbf{\Psi}[f](t) := \lim_{\delta \to 0} \iint\limits_{\Gamma \setminus \Gamma_{t,\delta}} |K(\zeta - t)|\, |\sigma(\zeta)|\, |f(\zeta) - f(t)|, \qquad t \in \Gamma,$$

where $\Gamma_{t,\delta} := \{\zeta \in \Gamma : |\zeta - t| \leqslant \delta\}$, *exists uniformly with respect to* $t \in \Gamma$. *Then the integral*

$$\mathbf{F}[f](t) := \lim_{\delta \to 0} \iint\limits_{\Gamma \setminus \Gamma_{t,\delta}} K(\zeta - t)\, \sigma(\zeta)\, (f(\zeta) - f(t)), \qquad t \in \Gamma,$$

and continuous extensions $\mathbf{\Phi}^+[f]$ *and* $\mathbf{\Phi}^-[f]$ *of the Cauchy-type integral* $\mathbf{\Phi}[f]$ *exist as well and the following formulas*

$$\mathbf{\Phi}^+[f]\,(t) = \mathbf{F}[f]\,(t) + f(t), \qquad t \in \Gamma, \tag{9}$$

$$\mathbf{\Phi}^-[f]\,(t) = \mathbf{F}[f]\,(t). \qquad t \in \Gamma,$$

hold true.

The function $\mathbf{\Phi}^+[f]\,(z)$ is hyperholomorphic in the domain Ω^+ by Theorem 3.1 and it is continuous on its closure $\overline{\Omega^+}$ by Theorem 3.2. Applying to it the formula (2.7), we obtain

$$\mathbf{\Phi}[\mathbf{\Phi}^+[f]]\,(z) = \mathbf{\Phi}^+[f](z), \quad z \in \Omega^+.$$

Therefore

$$\mathbf{\Phi}^+[\mathbf{\Phi}^+[f]]\,(t) = \mathbf{\Phi}^+[f](t), \quad t \in \Gamma,$$

and it follows from (9) that

$$\mathbf{\Phi}^+[f]\,(t) = \mathbf{F}[\mathbf{\Phi}^+[f]]\,(t) + \mathbf{\Phi}^+[f](t), \quad t \in \Gamma,$$

$$\mathbf{F}[\mathbf{\Phi}^+[f]]\,(t) = 0, \quad t \in \Gamma,$$

$$\mathbf{F}[\mathbf{F}[f]]\,(t) + \mathbf{F}[f]\,(t) = 0, \quad t \in \Gamma.$$

Operator representation of the latter formula has the form

$$\mathbf{F}^2 = -\mathbf{F}. \tag{10}$$

Supposing the function f complex-valued, represent the operator $\mathbf{F}$ in the form

$$\mathbf{F} = \mathbf{B} + \mathbf{C}\boldsymbol{i}_1 + \mathbf{D}\boldsymbol{i}_2 + \mathbf{E}\boldsymbol{i}_3, \tag{11}$$

where for $t \in \Gamma$

$$\mathbf{B}[f](t) := \frac{1}{4\pi} \int\limits_{\Gamma} \frac{f(\zeta) - f(t)}{|\zeta - t|^3} ((\zeta_1 - t_1) d\zeta_2 d\zeta_3 + (\zeta_2 - t_2) d\zeta_3 d\zeta_1 + (\zeta_3 - t_3) d\zeta_1 d\zeta_2),$$

$$\mathbf{C}[f](t) := -\frac{1}{4\pi} \int\limits_{\Gamma} \frac{f(\zeta) - f(t)}{|\zeta - t|^3} ((\zeta_2 - t_2) d\zeta_1 d\zeta_2 - (\zeta_3 - t_3) d\zeta_3 d\zeta_1),$$

$$\mathbf{D}[f](t) := -\frac{1}{4\pi}\int\limits_{\Gamma}\frac{f(\zeta)-f(t)}{|\zeta-t|^3}((\zeta_3-t_3)d\zeta_2 d\zeta_3-(\zeta_1-t_1)d\zeta_1 d\zeta_2),$$

$$\mathbf{E}[f](t) := -\frac{1}{4\pi}\int\limits_{\Gamma}\frac{f(\zeta)-f(t)}{|\zeta-t|^3}((\zeta_1-t_1)d\zeta_3 d\zeta_1-(\zeta_2-t_2)d\zeta_2 d\zeta_3),$$

$t := t_1\boldsymbol{i}_1 + t_2\boldsymbol{i}_2 + t_3\boldsymbol{i}_3$.

Substituting expression (11) into equality (10), we obtain the next system of operator equalities:

$$\begin{cases} \mathbf{B}^2-\mathbf{C}^2-\mathbf{D}^2-\mathbf{E}^2=-\mathbf{B}, \\ \mathbf{BC}+\mathbf{CB}+\mathbf{DE}-\mathbf{ED}=-\mathbf{C}, \\ \mathbf{BD}+\mathbf{DB}+\mathbf{EC}-\mathbf{CE}=-\mathbf{D}, \\ \mathbf{BE}+\mathbf{EB}+\mathbf{CD}-\mathbf{DC}=-\mathbf{E}. \end{cases} \tag{12}$$

It is known (see [12]), that Hölder spaces $H^\alpha(\Gamma)$, $\alpha \in (0,1)$, are invariant with respect to the operator $\mathbf{F}$, and therefore to the operators $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ and $\mathbf{E}$ as well.

Definition 3.3 ([13])**.** Let $\mathbf{T}$ be a linear operator in a complex Banach space. A number $\lambda \in \mathbb{C}$ is called an **eigenvalue** of the operator $\mathbf{T}$ if the equation

$$\mathbf{T}f = \lambda f$$

has a nonzero solution f which is called an **eigen element** of the operator $\mathbf{T}$. The set of eigenvalues λ is called the **point spectrum** of the operator $\mathbf{T}$. Eigen elements, corresponding to a fix eigenvalue λ, form so-called **eigen subspace**; its dimension is called the **multiplicity** of the eigenvalue λ.

Owing to operator equations (11), (12), the spectra of operators $\mathbf{F}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ and $\mathbf{E}$ are closely interrelated, which is confirmed by the following theorems.

Theorem 3.4. *Points* 0 *and* -1 *are eigenvalues of infinity multiplicity of the operator* $\mathbf{F}$*.*

Proof. It follows from formulas (9) and (2.7), that boundary value of any function, hyperholomorphic in the domain Ω^+ and continuous on its closure, is an eigen element of the operator $\mathbf{F}$, corresponding to the eigen value 0. And from formula (2.8) it follows that eigen element, corresponding to the eigen value -1, is the boundary value on the surface Γ of any function, hyperholomorphic in the domain Ω^- and continuous on its closure and tending to zero in infinity. Theorem is proved. □

Theorem 3.5. *Point* 0 *is eigenvalue of operators* $\mathbf{B}$*,* $\mathbf{C}$*,* $\mathbf{D}$ *and* $\mathbf{E}$*.*

Proof. It follows from the equality (11), that any complex constant is an eigen element corresponding to the eigen value 0 of the operators $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ and $\mathbf{E}$. Theorem is proved. □

Let us confine ourselves to a class of functions f, satisfying the condition $\mathbf{B}[f](t) \equiv 0$, $t \in \Gamma$. Then $\mathbf{F}$ becomes purely vector operator and the system (12) gets the look

$$\begin{cases} \mathbf{C}^2 + \mathbf{D}^2 + \mathbf{E}^2 = \mathbf{0}, \\ \mathbf{DE} - \mathbf{ED} = -\mathbf{C}, \\ \mathbf{EC} - \mathbf{CE} = -\mathbf{D}, \\ \mathbf{CD} - \mathbf{DC} = -\mathbf{E}. \end{cases} \tag{13}$$

Theorem 3.6. *If λ is a eigenvalue of the operator $\mathbf{C}$, $\mathbf{D}$ or $\mathbf{E}$, then $\lambda \pm i$ are its eigenvalues too.*

Proof. For symmetry reasons it is enough to consider one from the three operators, for example, $\mathbf{C}$. It follows from the equality $\mathbf{C}\varphi = \lambda\varphi$ and from the third and fourth equalities of the system (13) that $(\mathbf{C} - \lambda\mathbf{I})^2\varphi^* = -\varphi^*$, where $\varphi^* := \mathbf{E}\varphi$. Therefore

$$\begin{cases} (\mathbf{C} - (\lambda + i)\mathbf{I})\,(\mathbf{C} - (\lambda - i)\mathbf{I})\,\varphi^* = 0, \\ (\mathbf{C} - (\lambda - i)\mathbf{I})\,(\mathbf{C} - (\lambda + i)\mathbf{I})\,\varphi^* = 0. \end{cases} \tag{14}$$

And it follows from the first equality of the system (14) that $\mathbf{C}\varphi^{**} = (\lambda + i)\,\varphi^{**}$, where $\varphi^{**} := (\mathbf{C} - (\lambda - i)\mathbf{I})\ \varphi^*$ is eigen element of the operator $\mathbf{C}$ corresponding to its eigenvalue $\lambda + i$. In the same way the second equality of the system (14) gives $\mathbf{C}\varphi^{**} = (\lambda - i)\,\varphi^{**}$, where $\varphi^{**} := (\mathbf{C} - (\lambda + i)\mathbf{I})\ \varphi^*$ is eigen element of the operator $\mathbf{C}$ corresponding to its eigenvalue $\lambda - i$. Theorem is proved. □

Note that the function φ^* is not unique. For operator $\mathbf{C}$ it can also be the function $\mathbf{D}\varphi$.

Theorem 3.7. *If any two of operators $\mathbf{C}$, $\mathbf{D}$, $\mathbf{E}$ have a common eigen element, then it is an eigen element for third operator too and it corresponds to their common eigenvalue* 0*.*

Proof. For symmetry reasons it is enough to consider only one pair of operators, for example $\mathbf{C}$ and $\mathbf{D}$. Let $\mathbf{C}\varphi = \lambda_1\varphi$, $\mathbf{D}\varphi = \lambda_2\varphi$. It follows from the fourth equality of the system (13) that $\mathbf{E}\varphi = 0$ whereupon we obtain from the second and third equalities that $\mathbf{C}\varphi = 0$ and $\mathbf{D}\varphi = 0$. Theorem is proved. □

References

[1] O. F. Gerus, V. N. Kutrunov, M. Shapiro, *On the spectra of some integral operators related to the potential theory in the plane,* Mathematical Methods in the Applied Sciences **33** (2010), 1685–1691.

[2] O. F. Gerus, *On spectrum of the reduced singular integral Cauchy operator and its components,* Zb. Pr. Inst. Mat. NAN Ukr. **9**, no. 2 (2012), 87–94 (Russian).

[3] O. F. Gerus, M. V. Shapiro *On a Cauchy-type integral related to the Helmholtz operator in the plane,* Boletin de la Sociedad Matemática Mexicana **10**, no. 1 (2004), 63–82.

[4] B. V. Shabat, *Introduction to complex analysis. Part 1. Functions of one variable,* "Nauka", Moscow (1976) (Russian).

[5] A. S. Meylekhzon, *On monogenity of quaternions,* Doklady Acad. Nauk SSSR **59** (1948), 431–434.

[6] G. Moisil, N. Theodoresco, *Functions holomorphes dans l'espace,* Mathematica (Cluj) **5** (1931), 142–159.

[7] S. A. Plaksa, V. S. Shpakivskyi, *Cauchy theorem for a surface integral in commutative algebras,* Complex Variables and Elliptic Equations **59** (2014), 110–119.

[8] O. F. Herus, *On the Cauchy theorem for hyperholomorphic functions of spatial variable,* Journal of Mathematical Sciences **229** (2018), 1–6.

[9] P. Mattila, *Geometry of sets and measures in Euclidean spaces. Fractals and rectifiability,* Cambridge University Press (1995).

[10] V. V. Kravchenko, M. V. Shapiro *Integral representations for spatial models of mathematical physics,* Addison Wesley Longman, Pitman Research Notes in Mathematics Series **351** (1996).

[11] R. A. Blaya, J. B. Reyes, M. Shapiro *On the Laplasian vector fields theory in domains with rectifiable boundary,* Mathematical Methods in the Applied Sciences **29** (2006), 1861–1881.

[12] R. A. Blaya, J. B. Reyes, *Boundary value problems for quaternionic monogenic functions on non-smooth surfaces,* Adv. Appl. Clifford algebras **9** (1999), 1–22.

[13] V. Hudson, J. Pym, *Applications of functional analysis and operator theory,* Math. in Science and Engineering, London: Academic Press **146** (1980).

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O SPEKTRUM KWATERNIONOWEGO OSOBLIWEGO OPERATORA CAŁKOWEGO I JEGO SKŁADOWE NA POWIERZCHNIACH PRZESTRZENNYCH

S t r e s z c z e n i e

Badamy punktowe spektrum kwaternionowego osobliwego operatora całkowego i rzeczywiste osobliwe generatory całkowe, ktre są jego składowymi; operatory działają na przestrzeni Höldera $\mathbf{H}^{\alpha}(\Gamma)$, $\alpha \in (0,1)$, gdzie Γ jest domkniętą, kwadratowalną powierzchnią.

Słowa kluczowe: kwaternion, funkcja różniczkowalna, hiperholomorficzna funkcja, osobliwy całkowy operator Cauchy'ego, spektrum